Invertible Bloom Lookup Tables

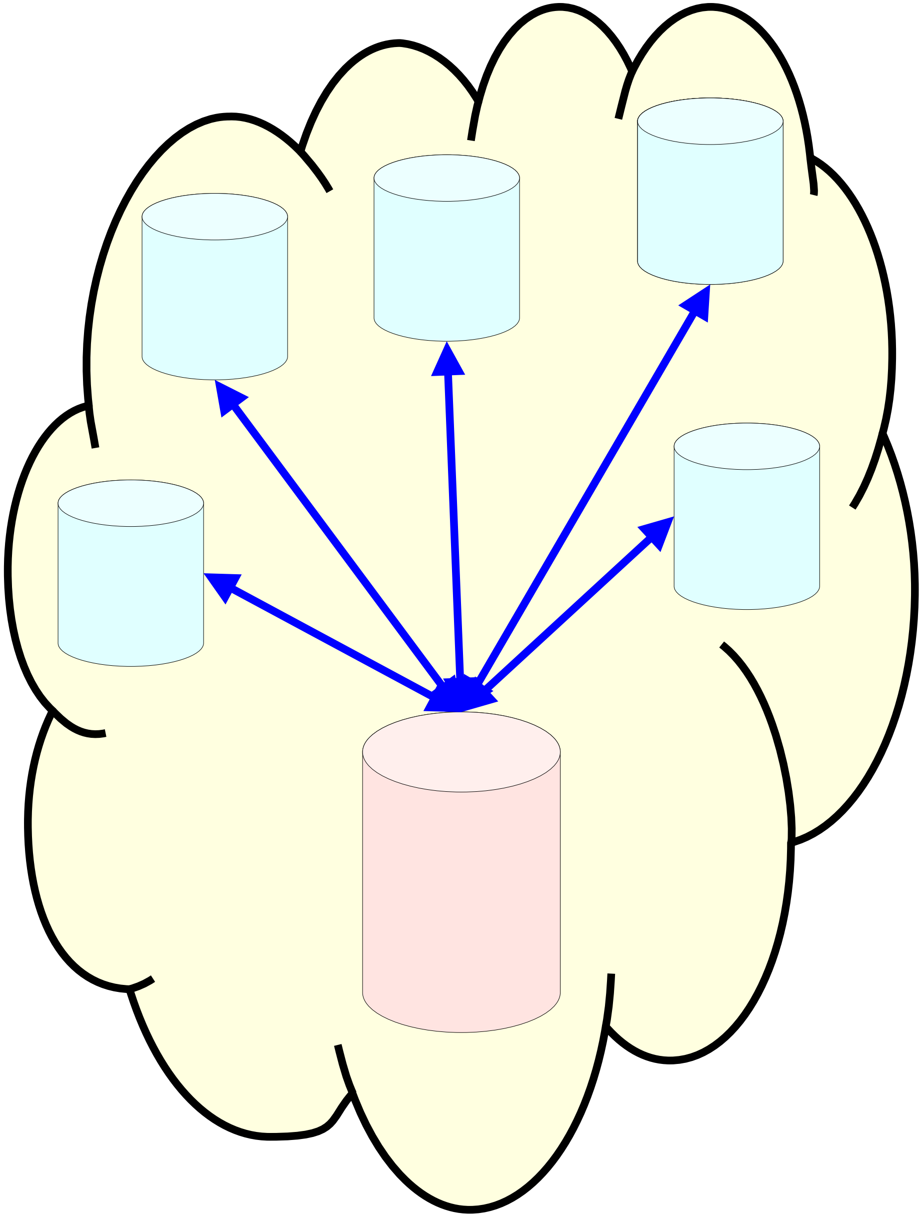
Outline for Today

- ***The Set Reconciliation Problem***
  - A problem in distributed systems.
- ***Invertible Bloom Lookup Tables***
  - A simple, fast, space-efficient solution to set reconciliation.
- ***Hypergraph Peeling***
  - An amazing technique for building data structures.
- ***Analyzing Hypergraph Peeling***
  - A *beautiful* analysis.

Motivation: ***Set Reconciliation***

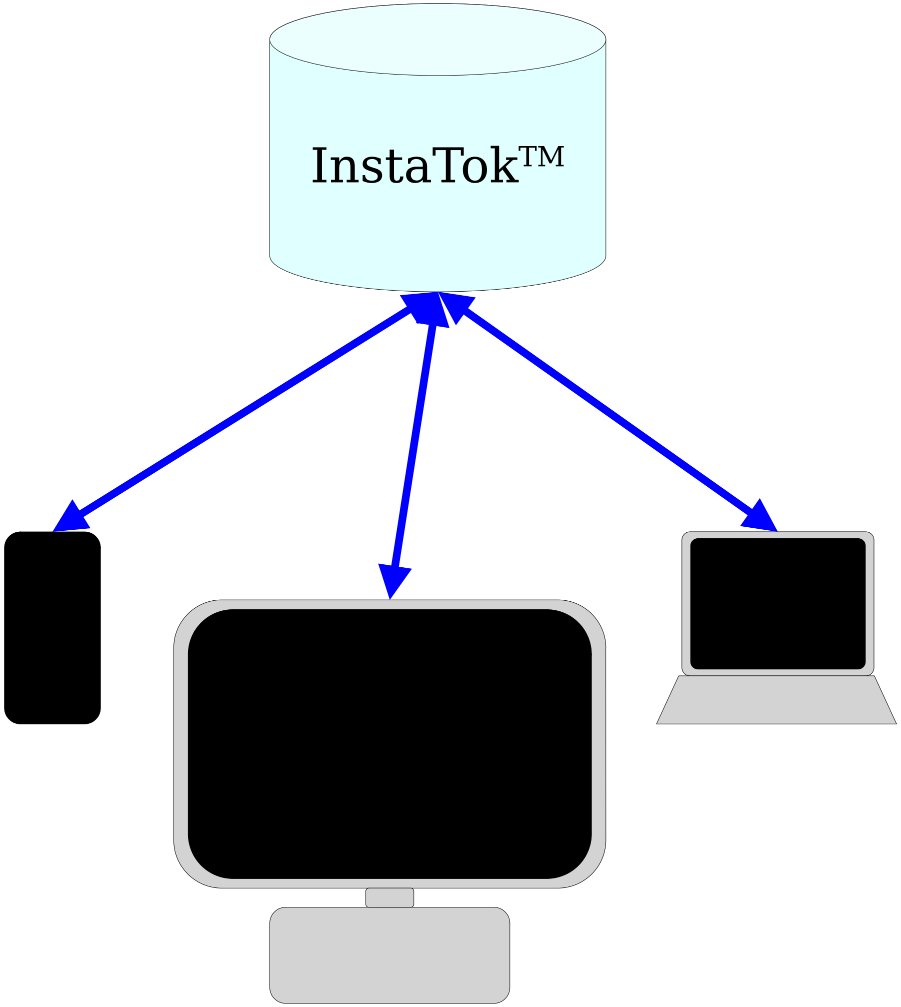
Three Motivating Problems

- We have a central backup server and multiple “frontier” servers.
- Updates happen on the frontiers, and need to be periodically synced to the backup server, but none of the frontier nodes know everything already on the server.
- ***Goal:*** Sync the backup with the frontier servers, using as little network bandwidth as possible.



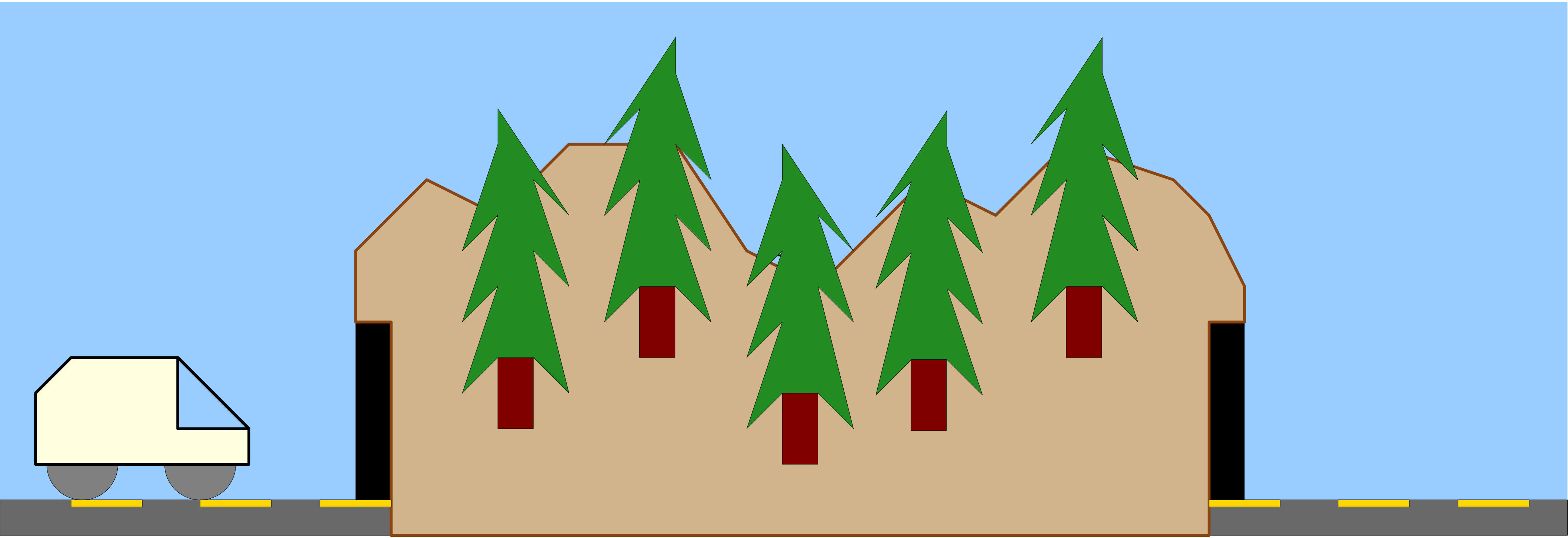
Three Motivating Problems

- Some services have multiple frontends (web, iOS, Android, etc.).
- Suppose you haven't logged into your account on one particular device in a while.
- You want to get updated copies of everything from the central server, but the server has no idea what data you already have.
- ***Goal:*** Do so, using as little network bandwidth as possible.



Three Motivating Problems

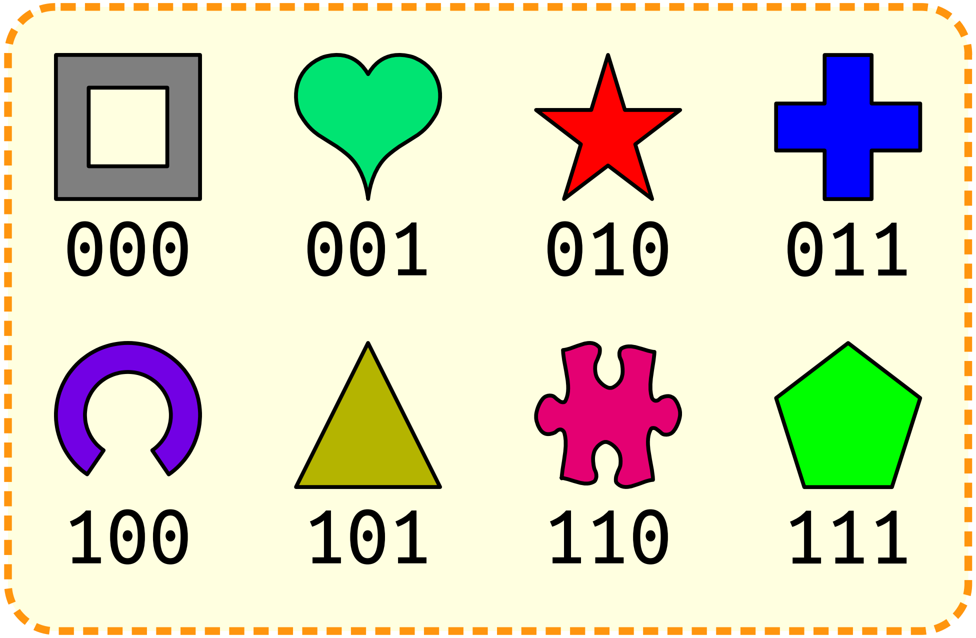
- You are designing the firmware for a modern car.
- The car was in the middle of downloading a software update when it lost signal (e.g. drove into a tunnel), and you don't know what data it received.
- ***Goal:*** Send the rest of the update to the car, using as little network bandwidth as possible.



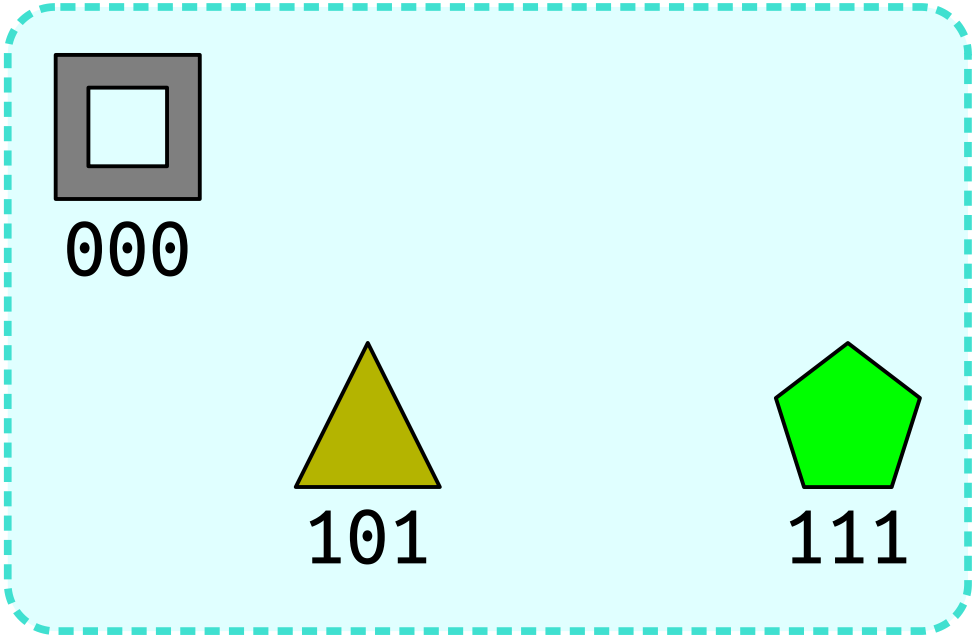
Set Reconciliation

- In the ***set reconciliation problem***, we have two parties each holding sets.
  - Anna holds a set A.
  - Bala holds a set $B \subseteq A$.
- Anna and Bala need to communicate with one another so that Bala ends up holding A.
- ***Goal:*** Solve this problem without requiring Anna and Bala to send “too much” information to each other.

Anna's Set A



We'll assume every item is represented as an integer of fixed size. (Say, the SHA-256 hash of the underlying data.)



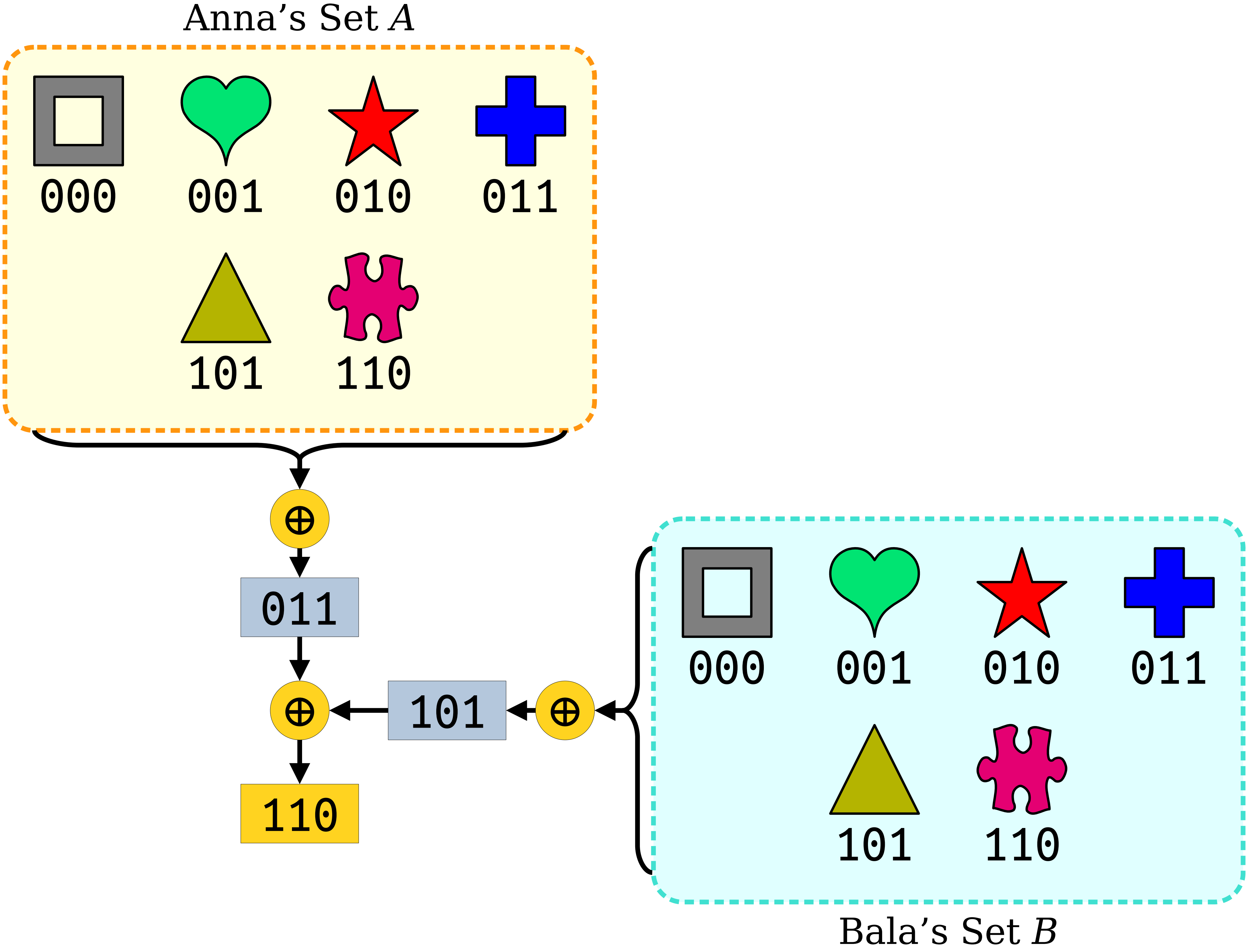
Bala's Set B

Quantifying our Approaches

- Let's let ***n*** = $|B|$ be the number of elements in Bala's set.
- Let's let ***k*** = $|A – B|$ be the number of elements Bala doesn't yet have.
- What's the space complexity of Anna sending everything to Bala?
  - ***Answer:*** $O(n + k)$.
- In practice, k will be much smaller than n.
- ***Goal:*** Solve this problem with space complexity $O(poly(k))$, with no dependency on n.
- How is this even possible?

Problem-Solving Technique: When a problem looks too hard, try solving a simpler version of it.

The k=1 Case


Anna's Set A
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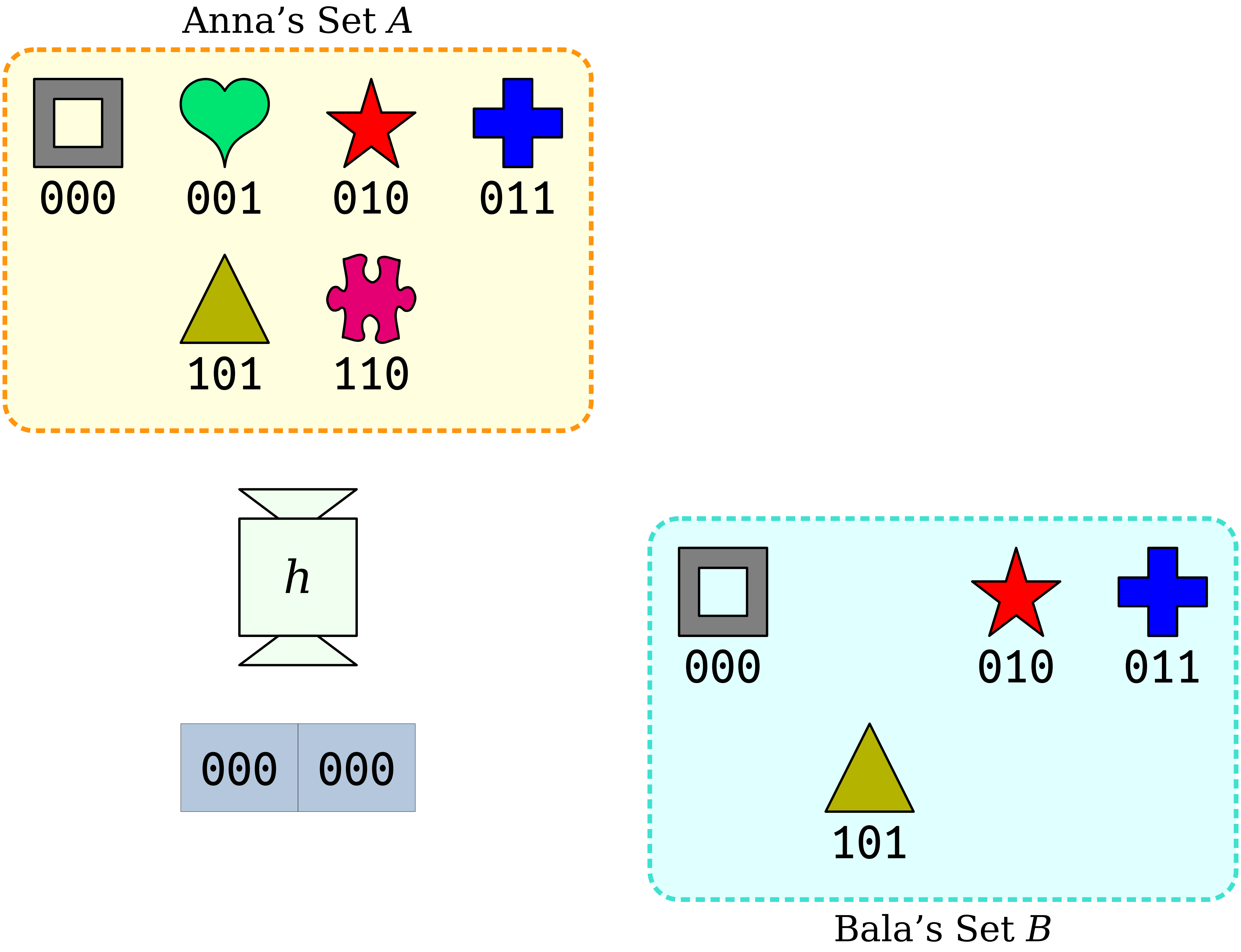
Bala's Set B

The k=2 Case


Anna’s Set A
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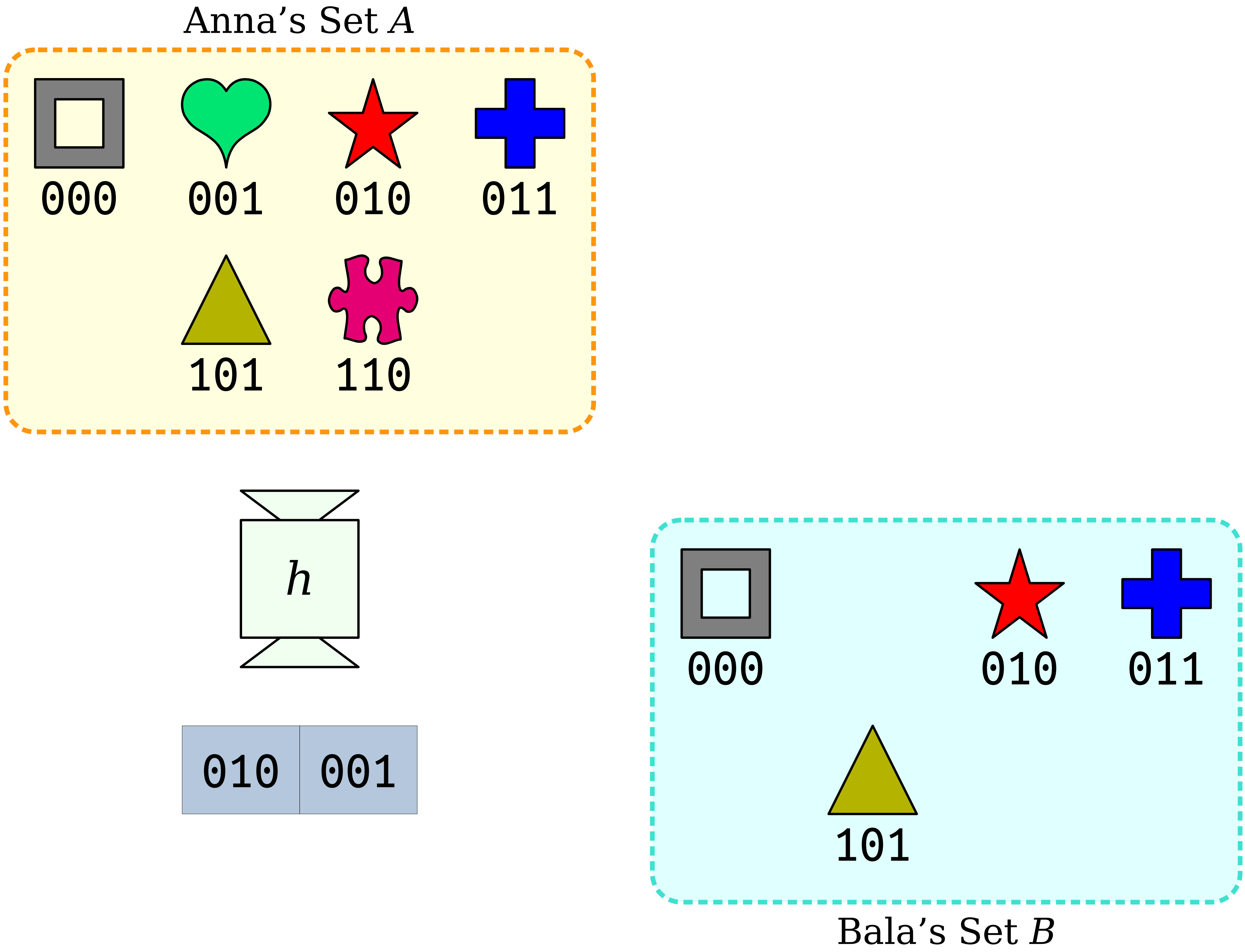
Bala’s Set B


Anna's Set A
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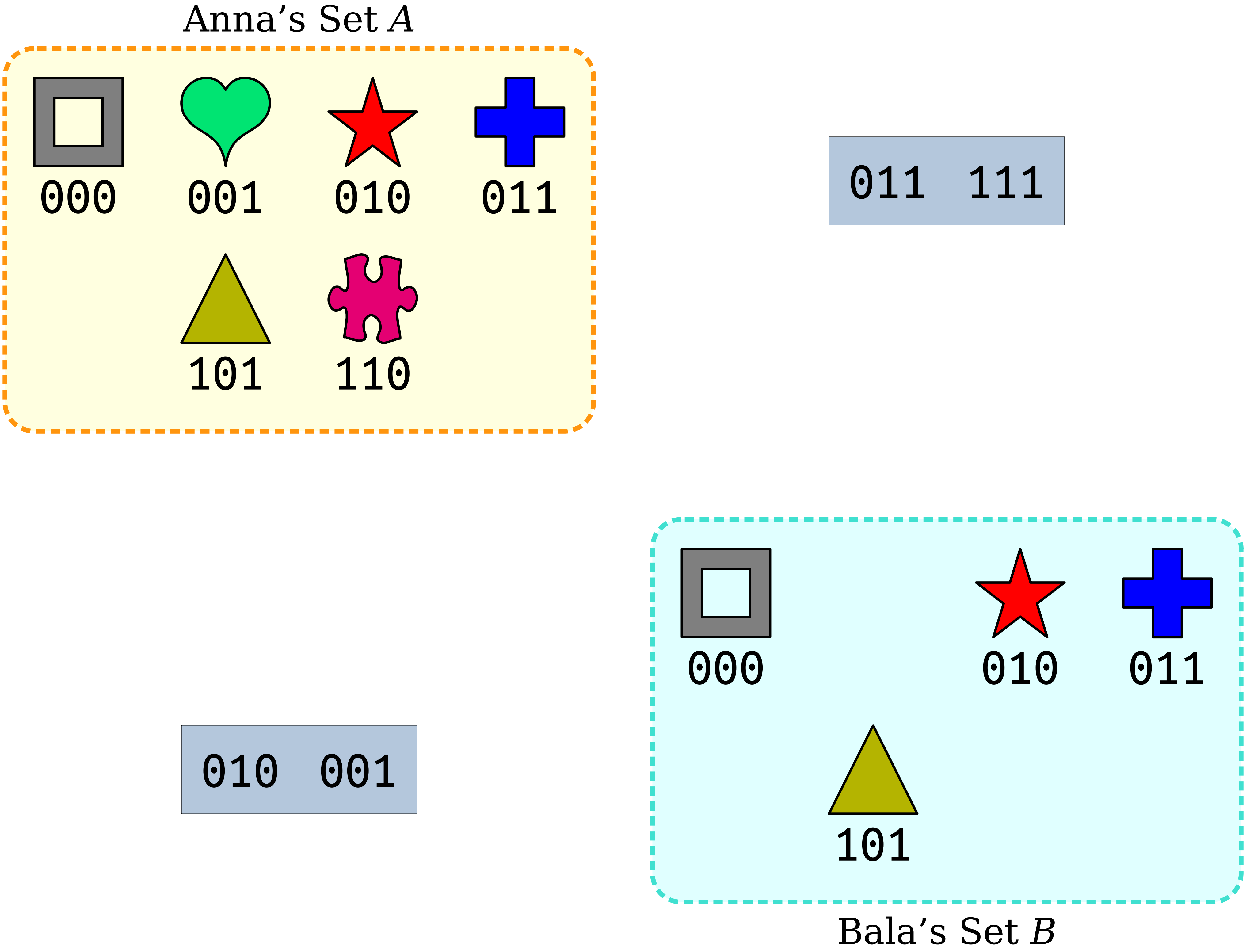
Bala's Set B


Anna’s Set A
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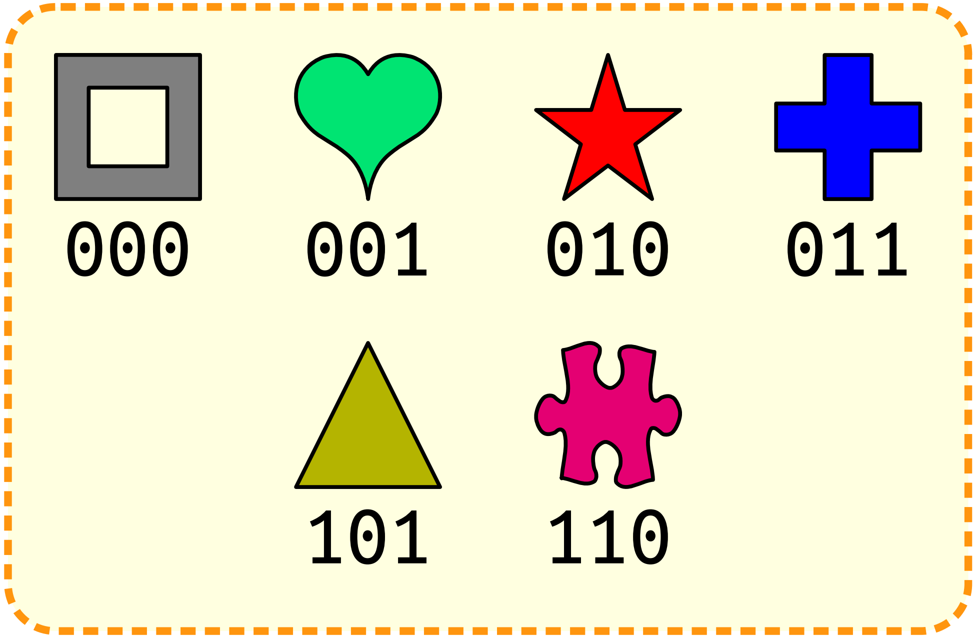
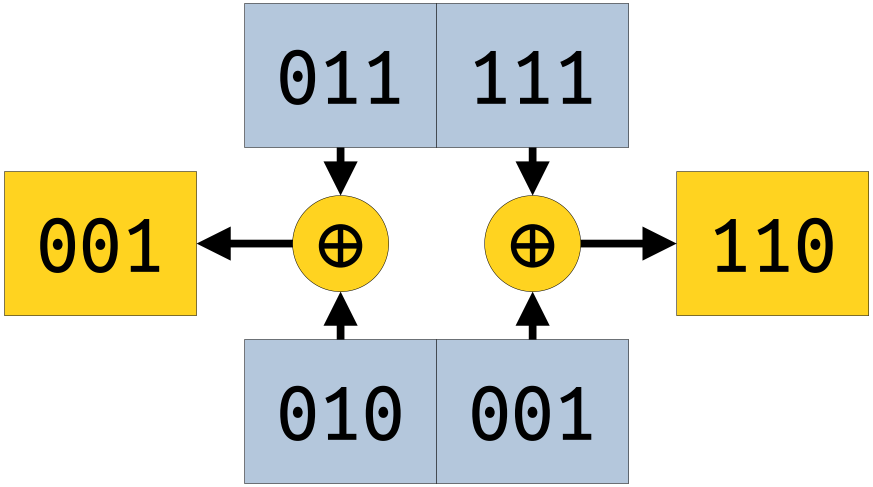
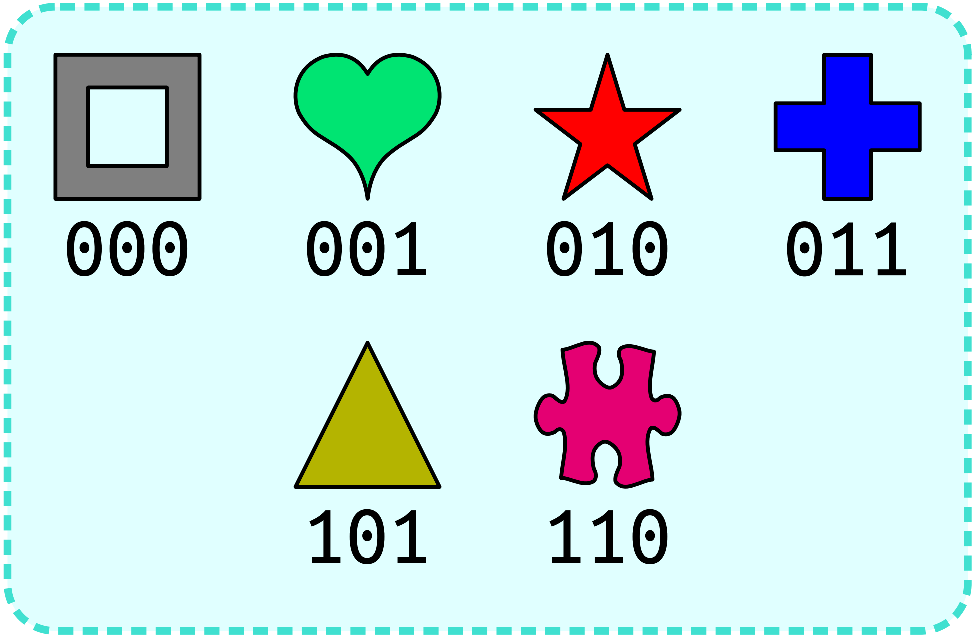
Bala’s Set B

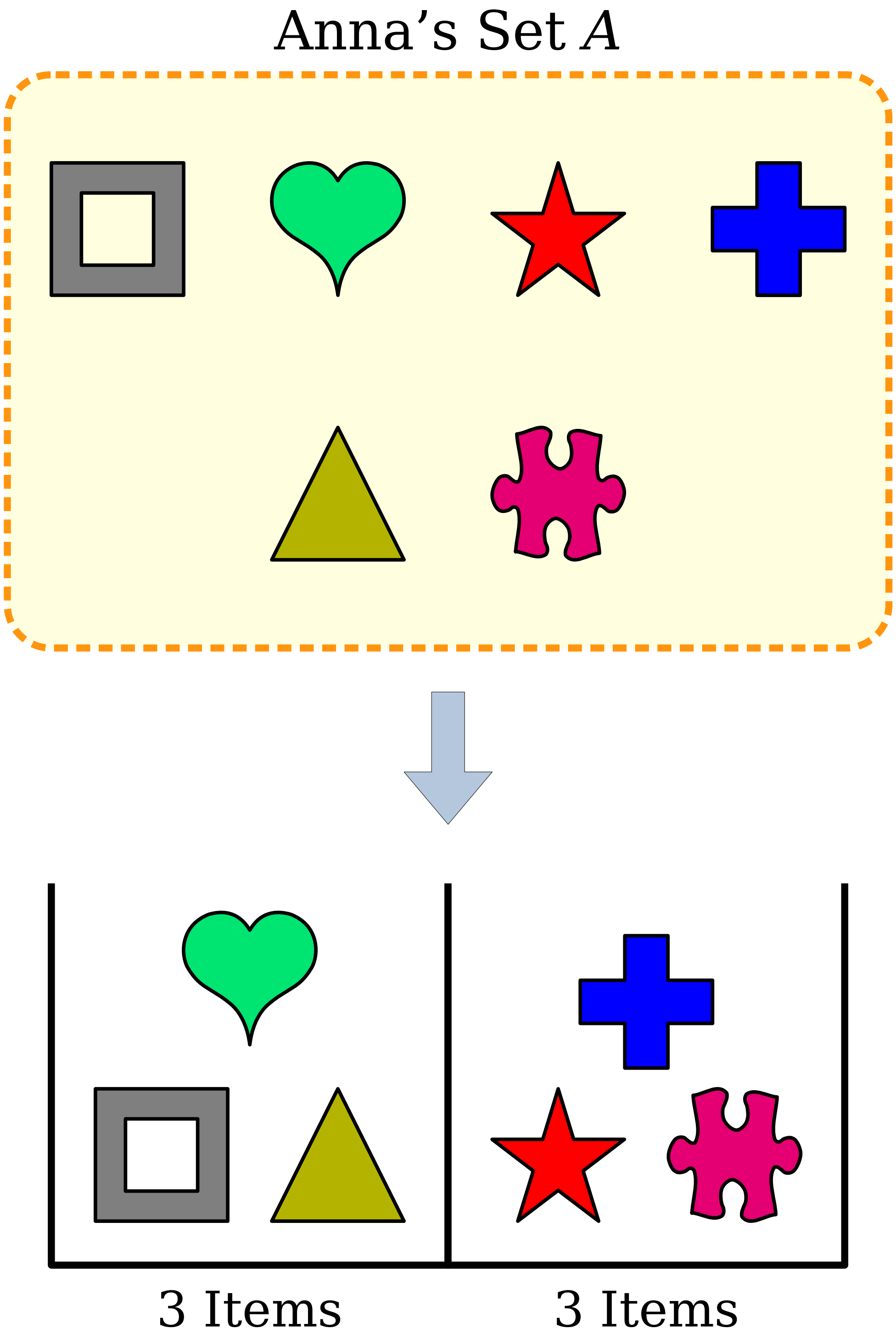
Anna’s Set *A*





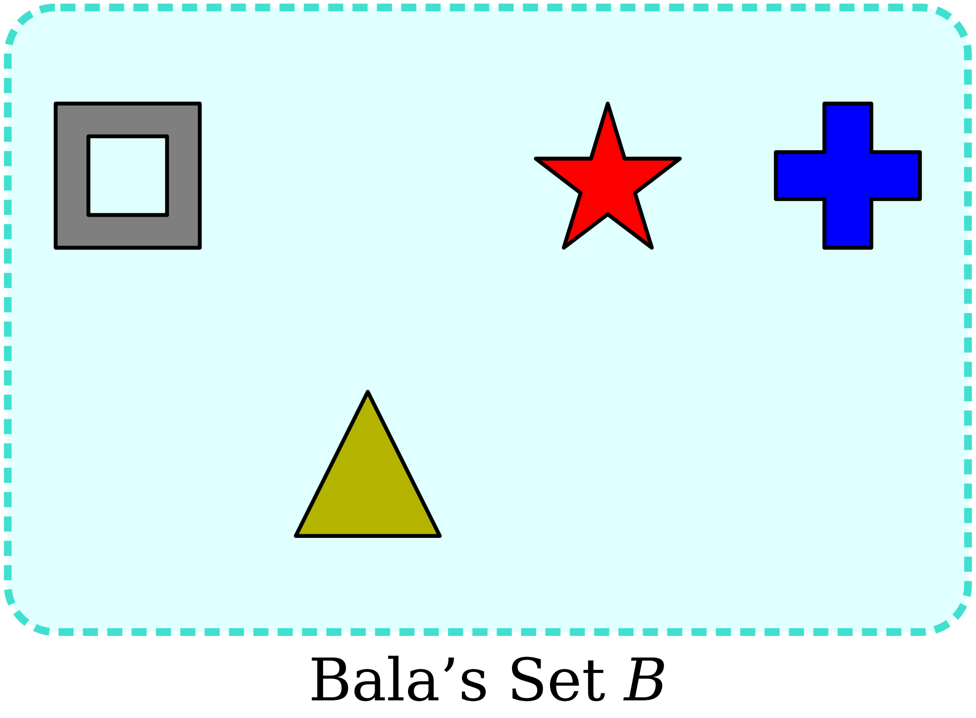


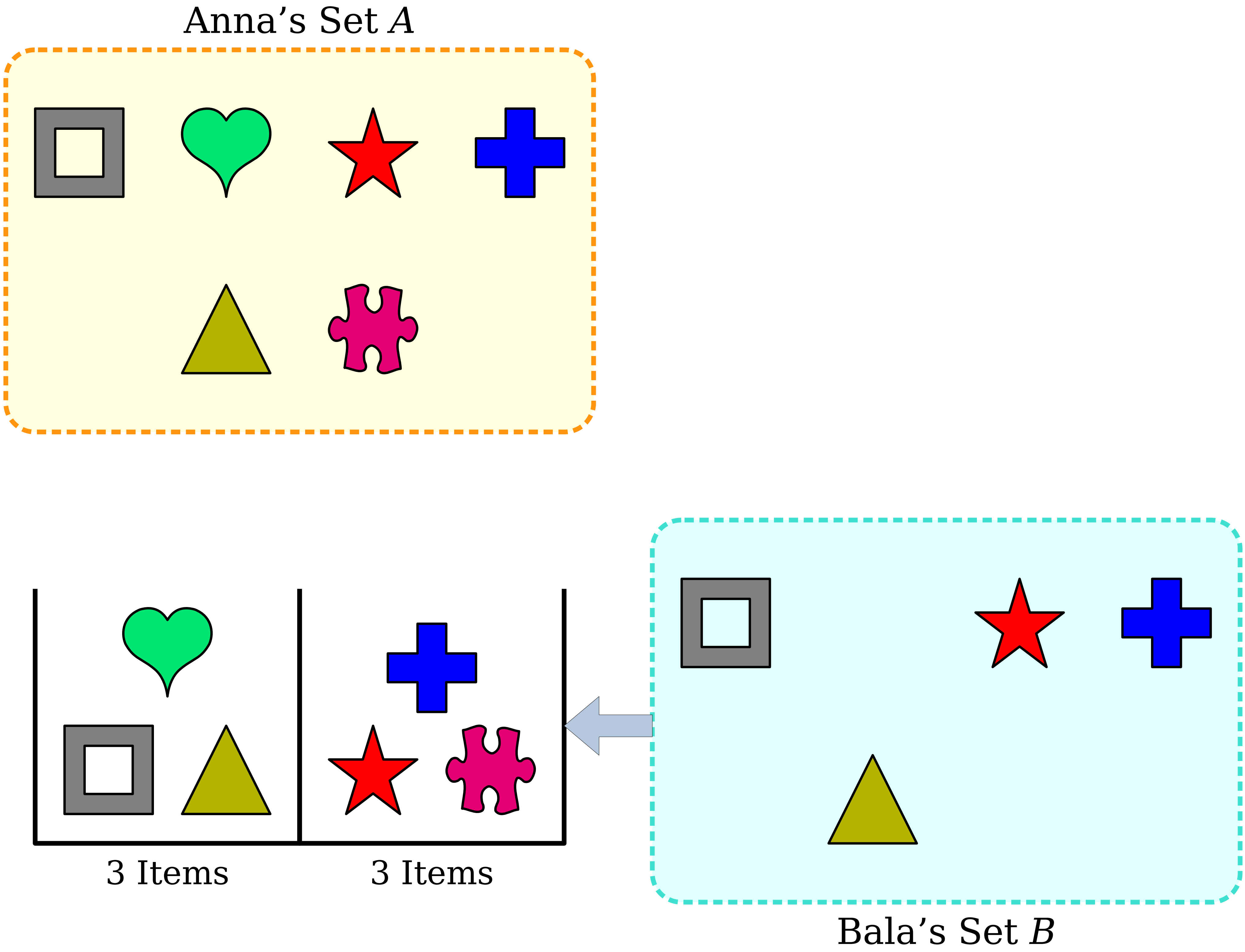
Bala’s Set *B*

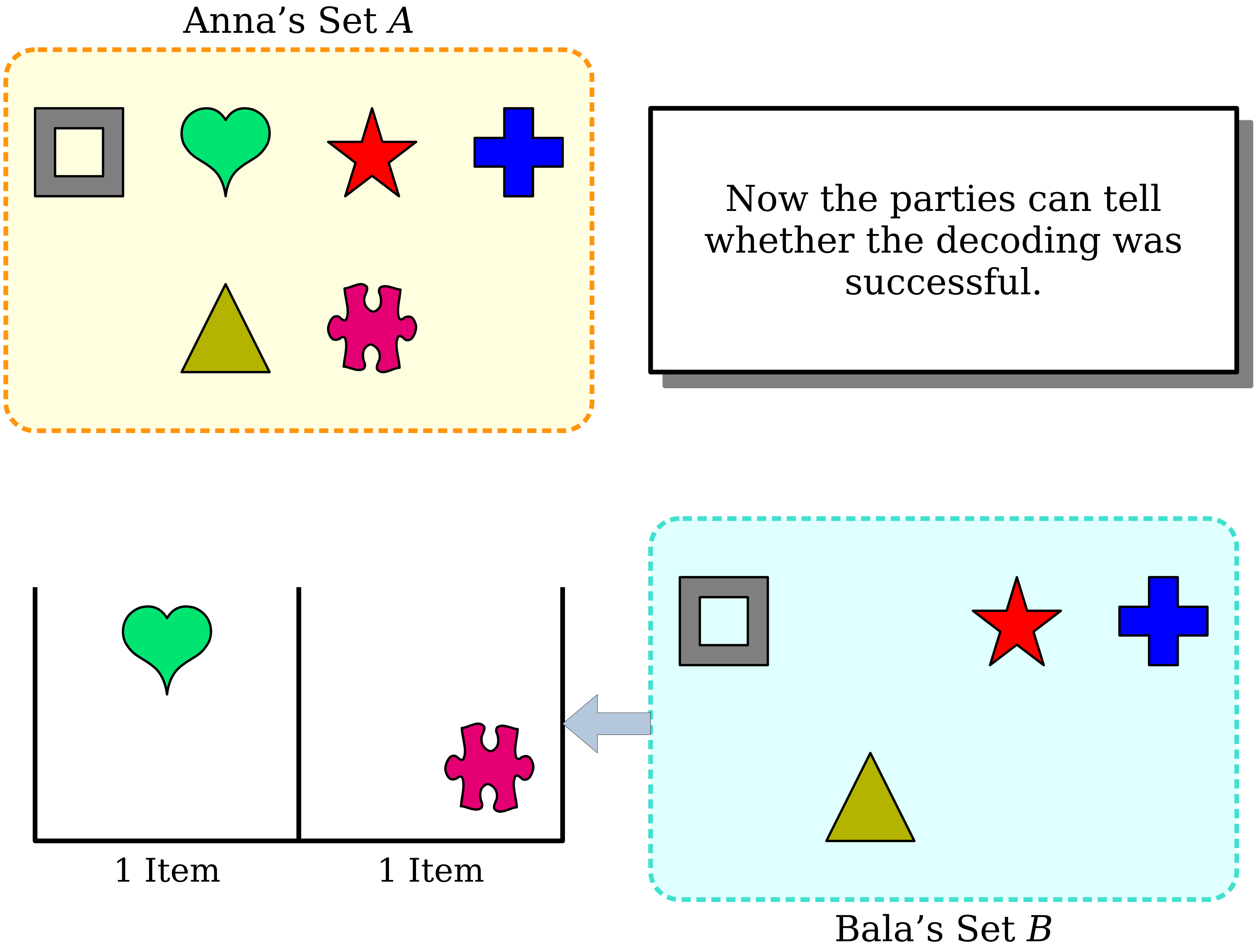


The “bucket of items” visually represents an XOR of items.

We can only tell what’s in the bucket if there’s exactly one item in it.




Anna’s Set A
3 Items
3 Items
Bala’s Set B



Anna's Set A

Now the parties can tell whether the decoding was successful.

1 Item

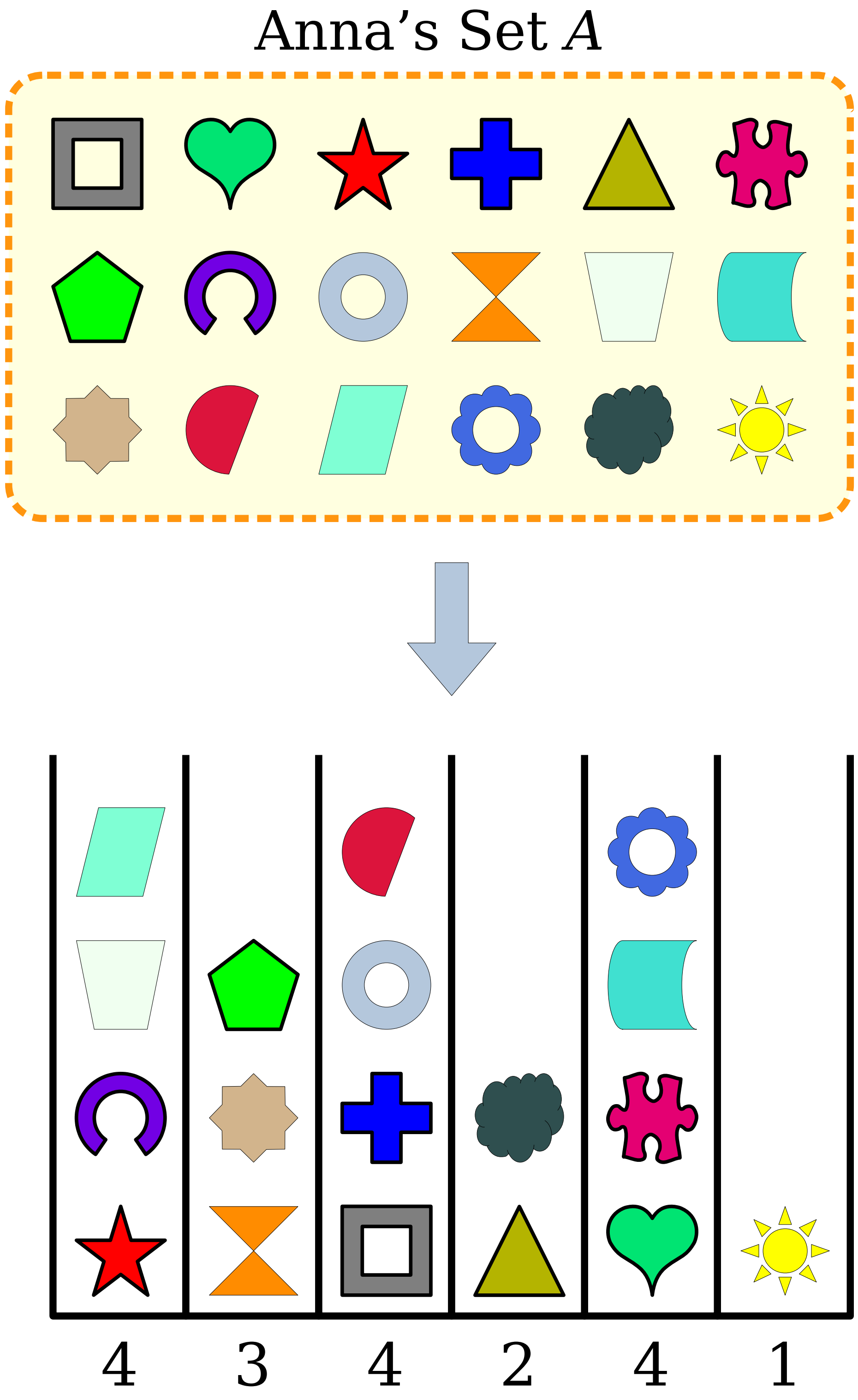
1 Item

Bala's Set B

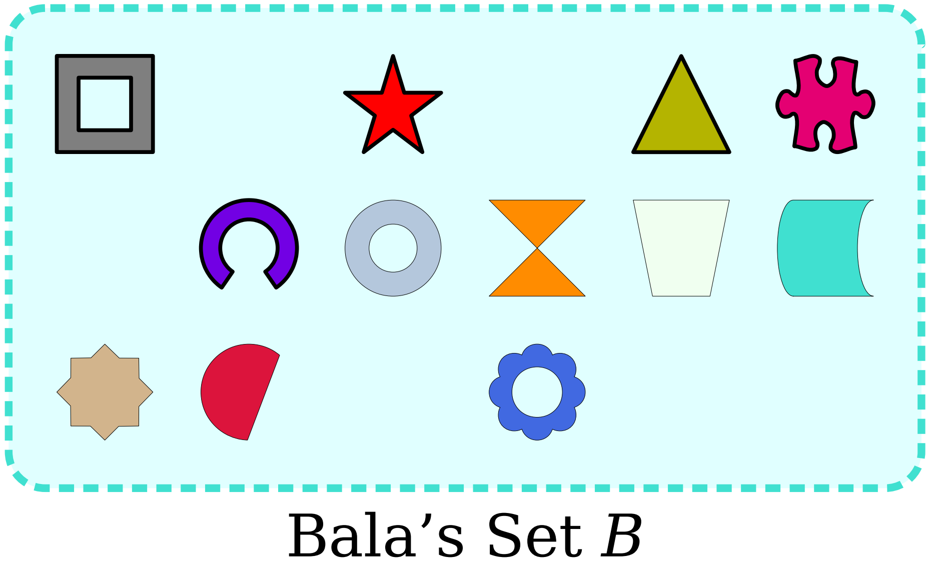
The k=2 Case

- This approach has a 50% chance of success.
    - (There are two ways to assign the items to distinct buckets and four possible assignments.)
- We can boost the success probability by replicating this approach multiple times with independent hash functions.
- With lg δ^{-1} replicated copies, we boost the success probability to $1 - \delta$.
- ***Question:*** Does this generalize to $k \geq 2$?

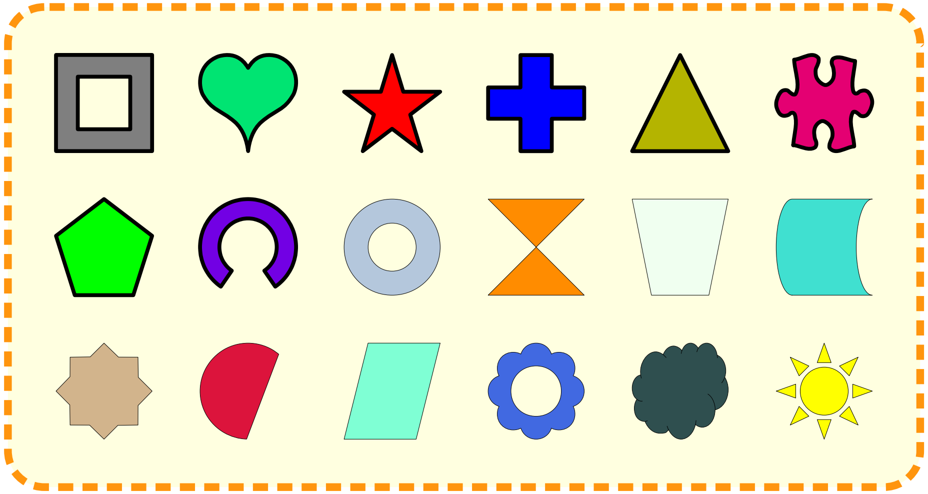
The General Case



Assume Anna and Bala know what k is.
(How might they do that?)



Anna's Set *A*



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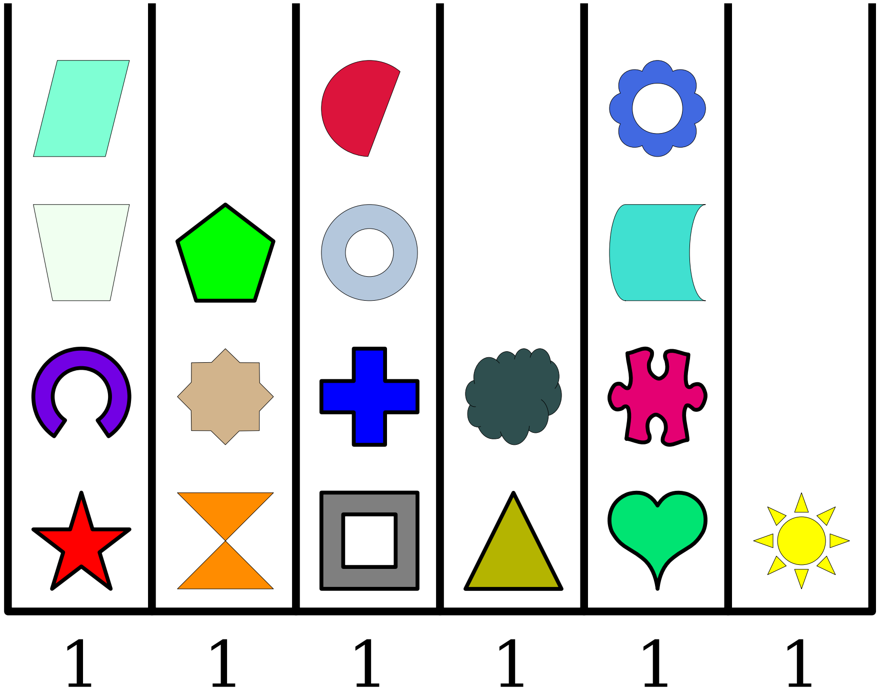
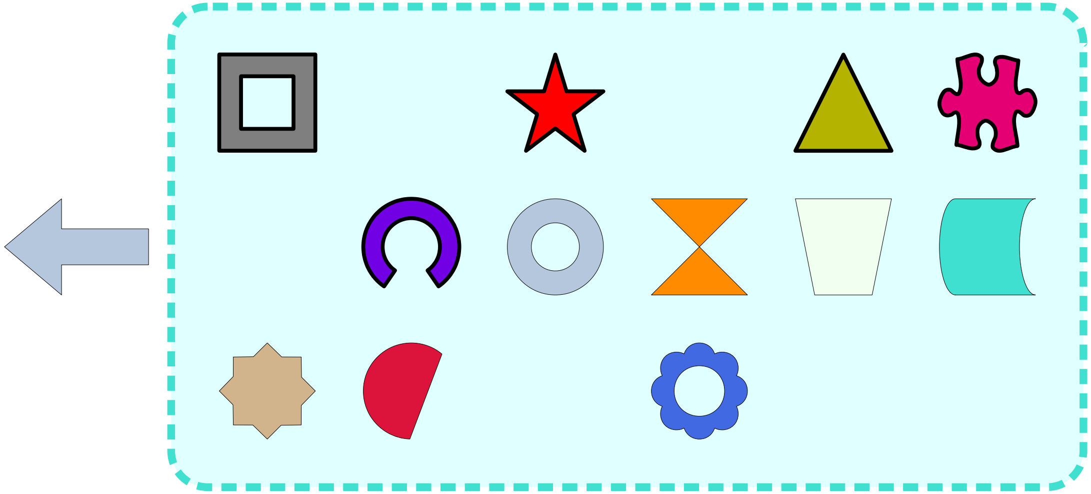
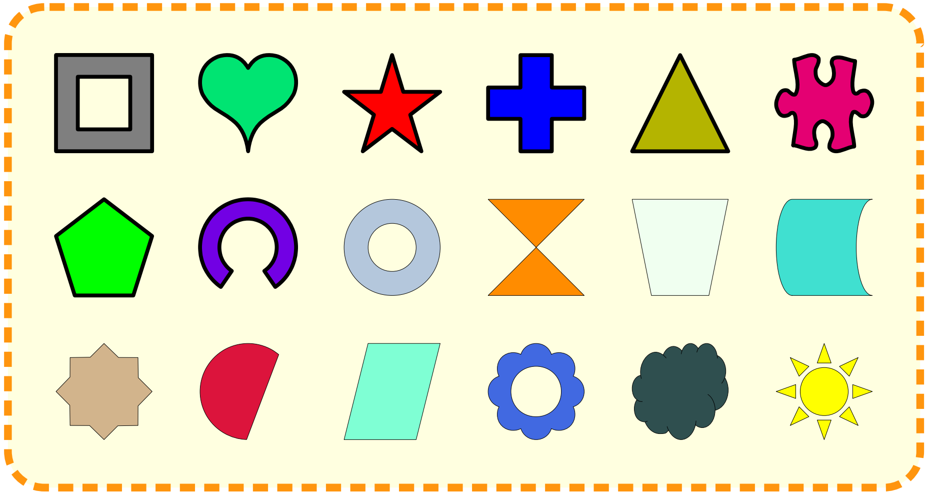
(How might they do that?)

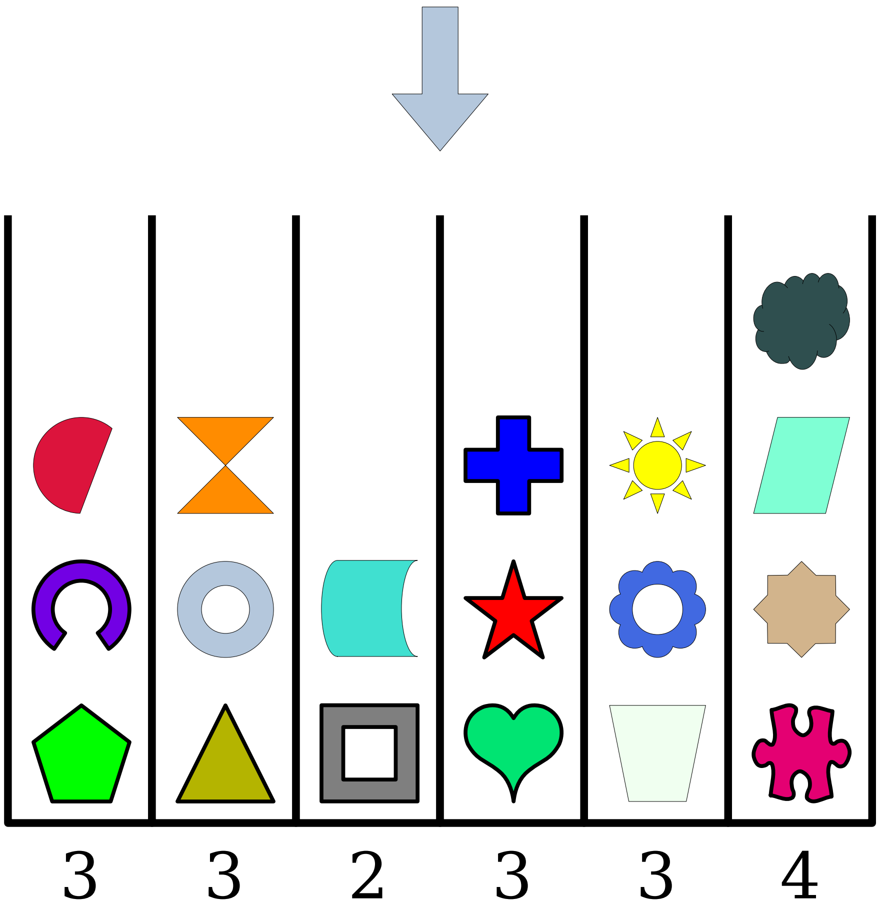




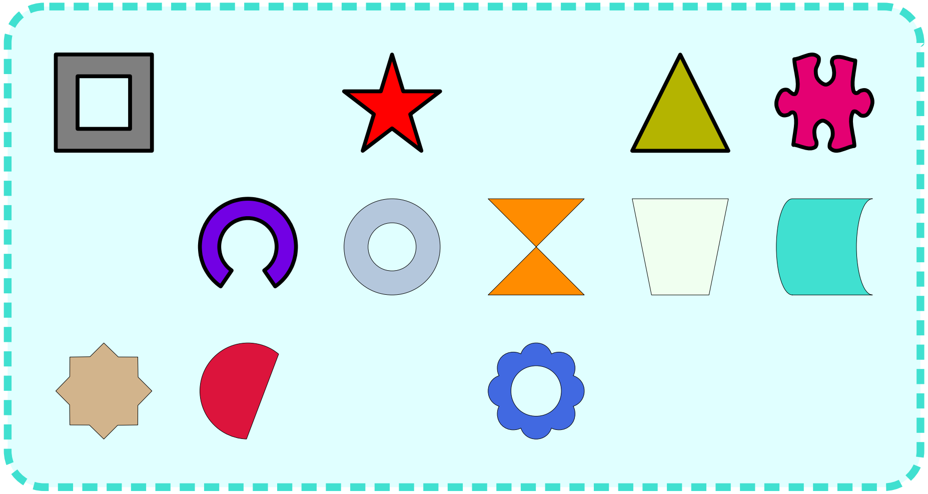
Bala's Set *B*

Anna’s Set A



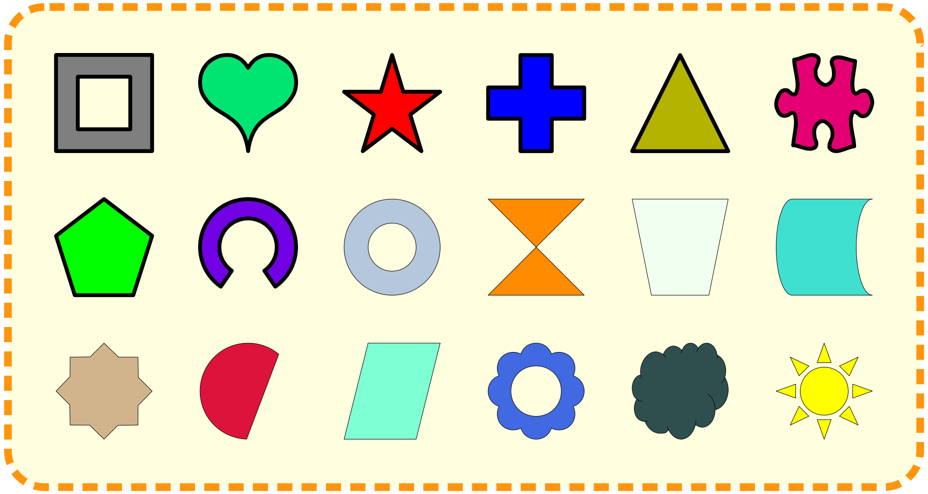


Assume Anna and Bala know what k is.
(How might they do that?)

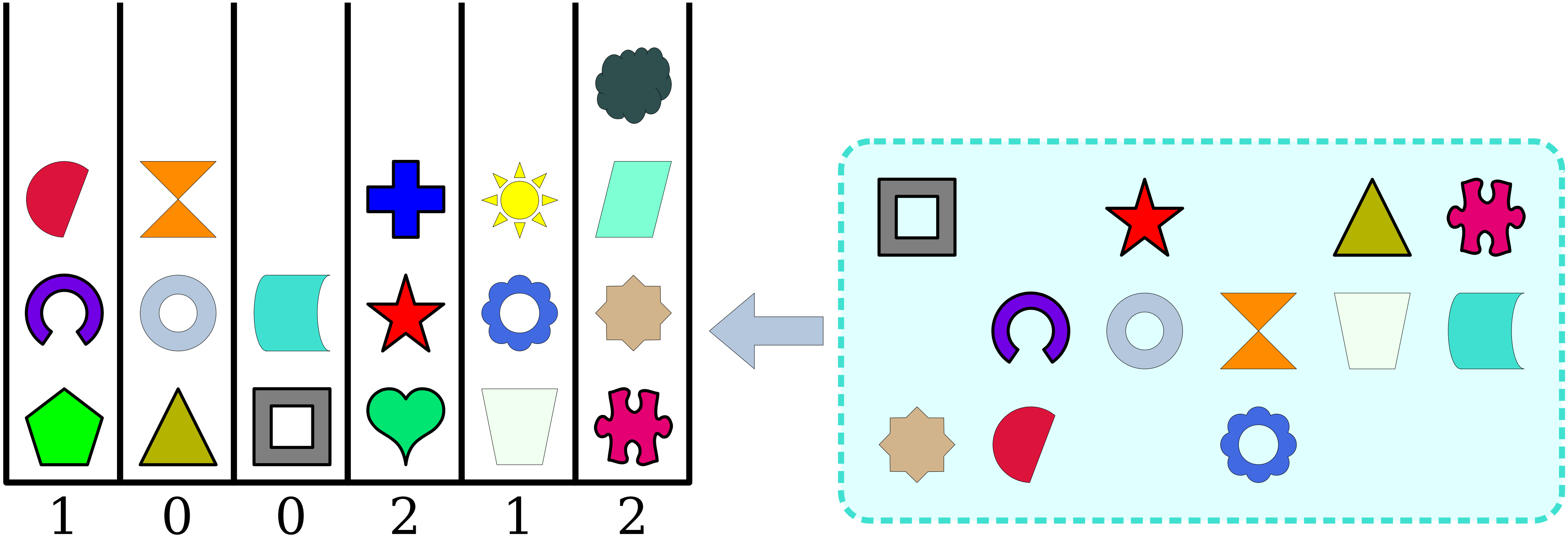


Bala’s Set B

Anna’s Set A



Assume Anna and Bala know what k is.
(How might they do that?)



Bala’s Set B

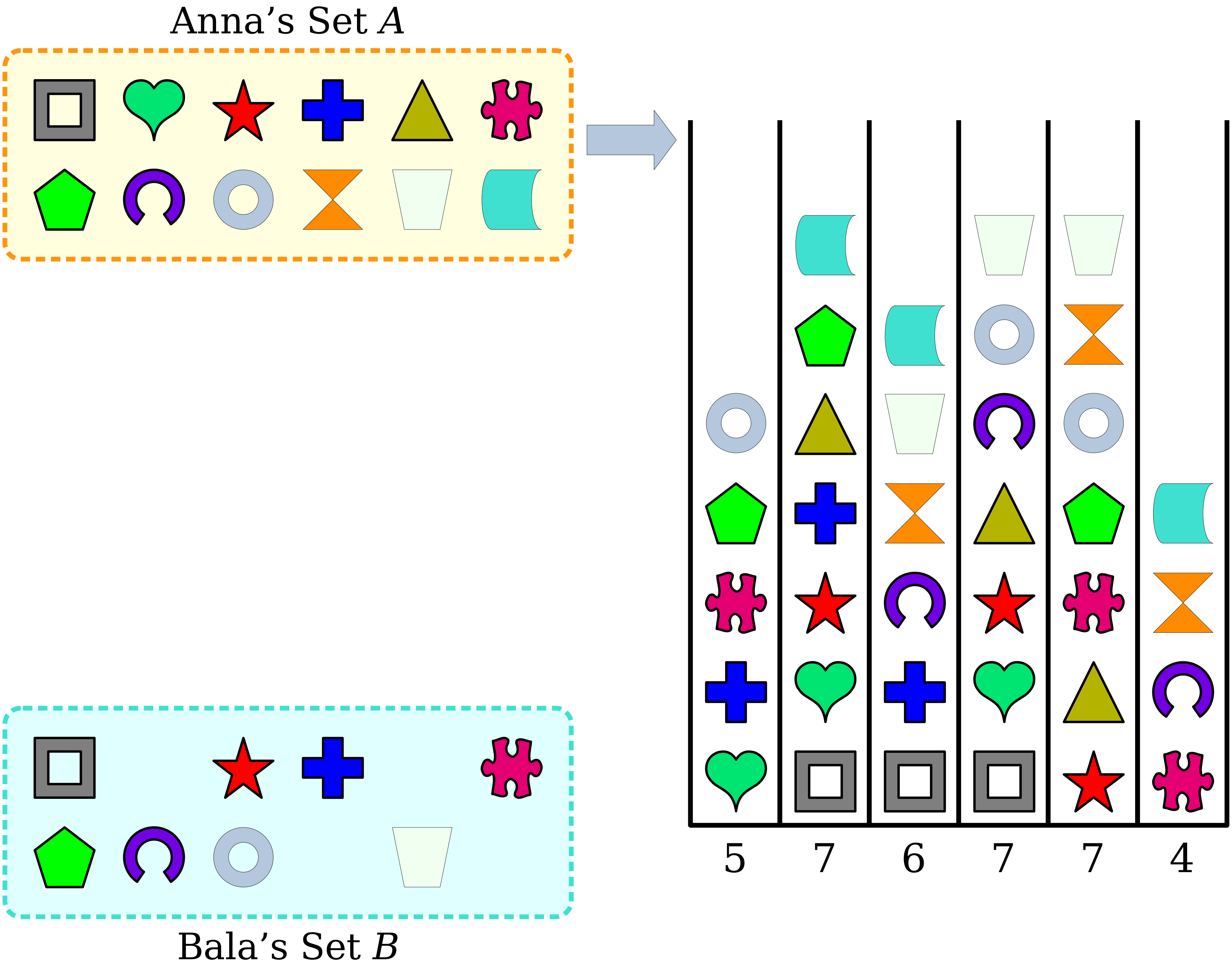
The General Case

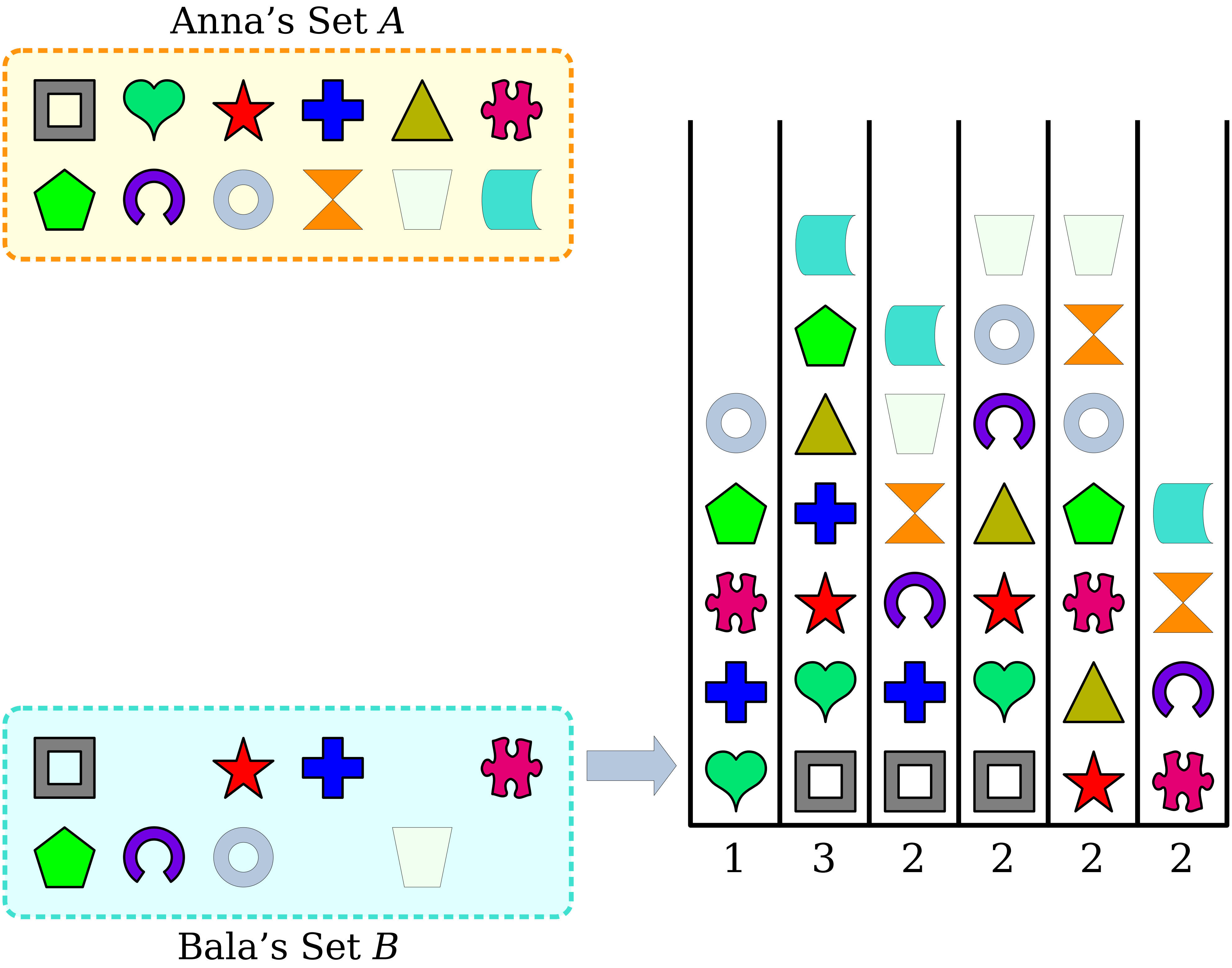
- Create an array of ***m*** buckets, all initially zero. Choose a hash function from elements to the set {0, 1, …, *m* – 1}.
- Anna hashes her items and adds them to the appropriate buckets; Bala hashes his items and removes them.
- Anna and Bala can identify all items in buckets whose counts are equal to one.
- This can be done without too much network communication.
  - Picking $m = k$ and replicating this 2 ln *k* times gives probability 1 – O(k^{-1}) of recovering all items and requires **2*k* ln *k*** buckets to be transmitted.
  - Picking $m = k$, recovering as many items as possible, then repeating this process on the remaining elements requires roughly **3.72*k*** buckets to be transmitted (on expectation, with high probability), but requires a lot of computation by both parties and multiple separate transmissions.
- ***Claim:*** We can do substantially better than this.

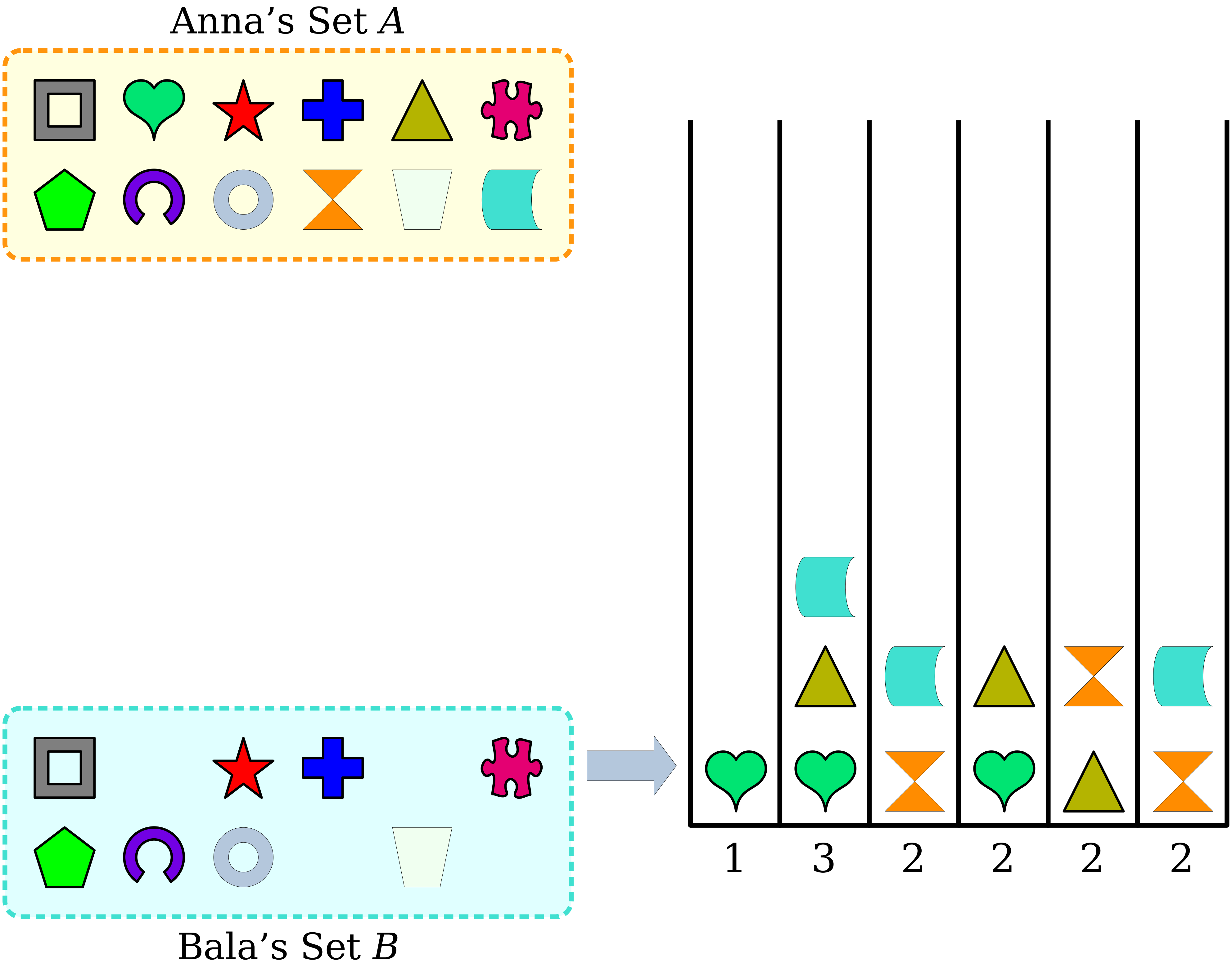
Taking a Step Back

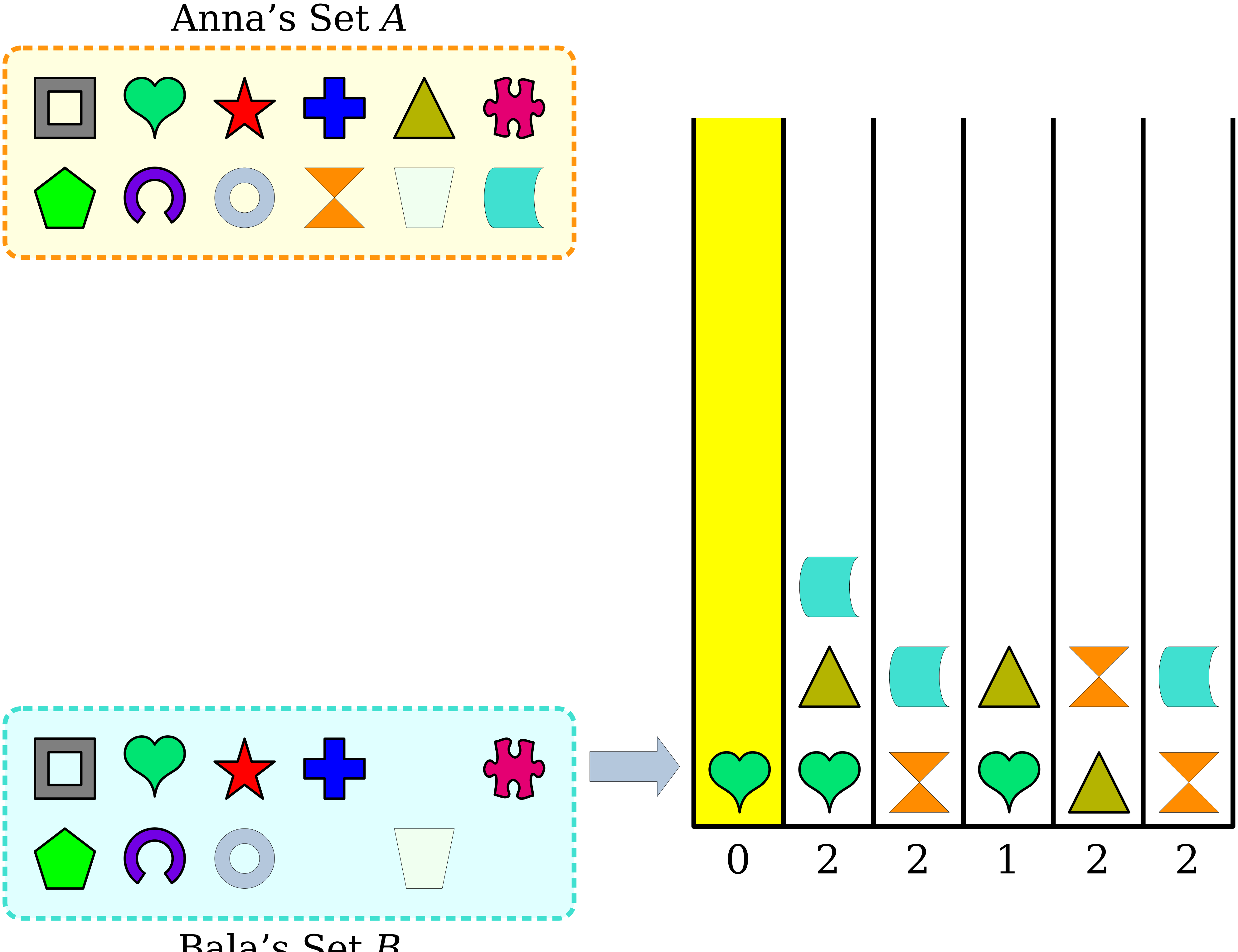
- We’re trying to find a way to hash items to buckets that doesn’t involve collisions.
- If you squint at this problem in just the right way, this kinda sorta ish looks like what cuckoo hashing was designed to solve.
- There are some major differences, though:
  - Our available memory is *way* smaller than the number of items.
  - We only care about collisions between *missing* items, and we don’t know what those are up front.
- ***Question:*** Are there any ideas we could adapt that would work here?

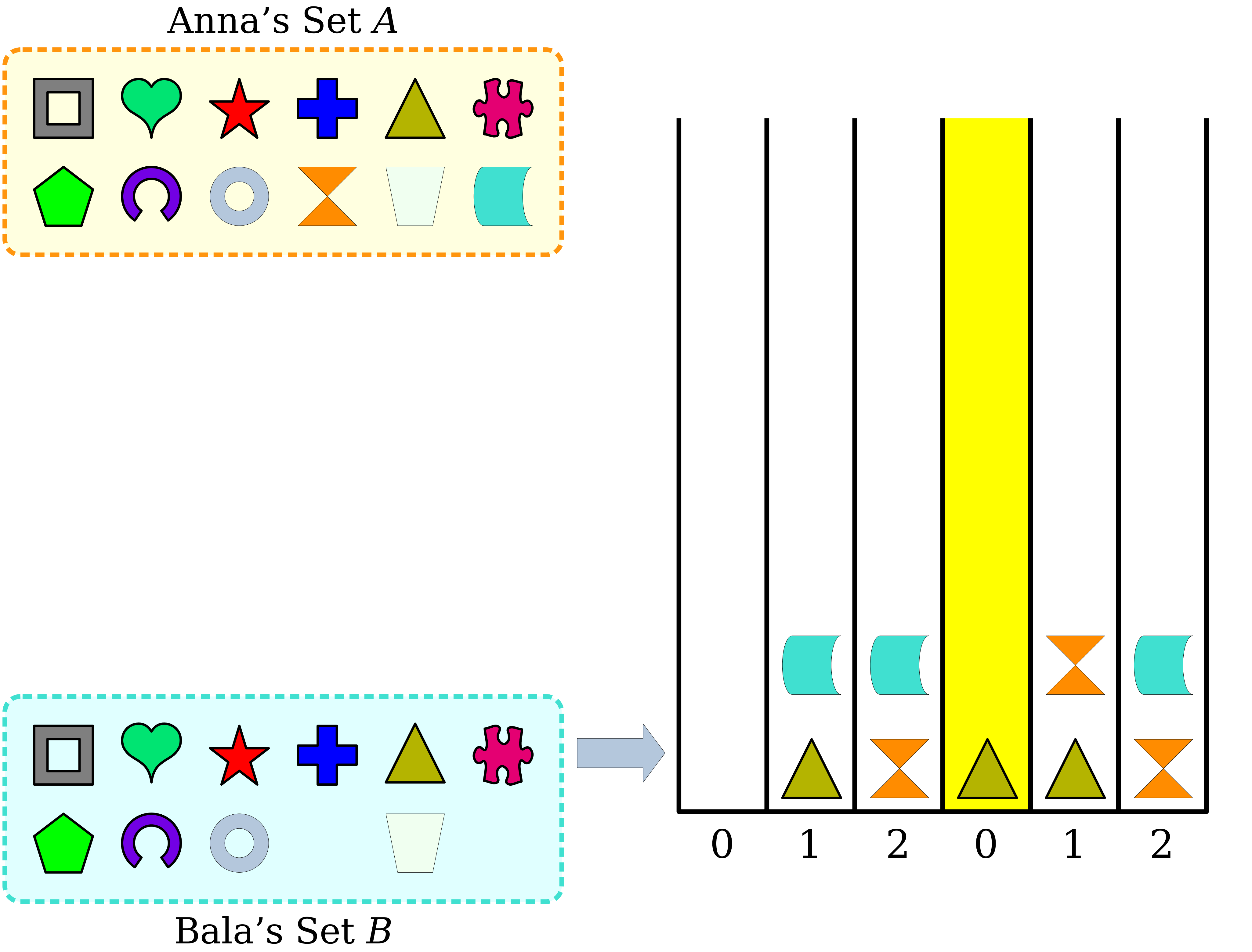
Idea: Hash each item to $d \geq 2$ buckets.


Anna's Set A
Bala's Set B
5
7
6
7
7
4


Anna’s Set A
Bala’s Set B
1
3
2
2
2
2


Anna’s Set A
Bala’s Set B
1
3
2
2
2
2


Anna’s Set A
Bala’s Set B
0
2
2
1
2
2


Anna’s Set A
Bala’s Set B
0
1
2
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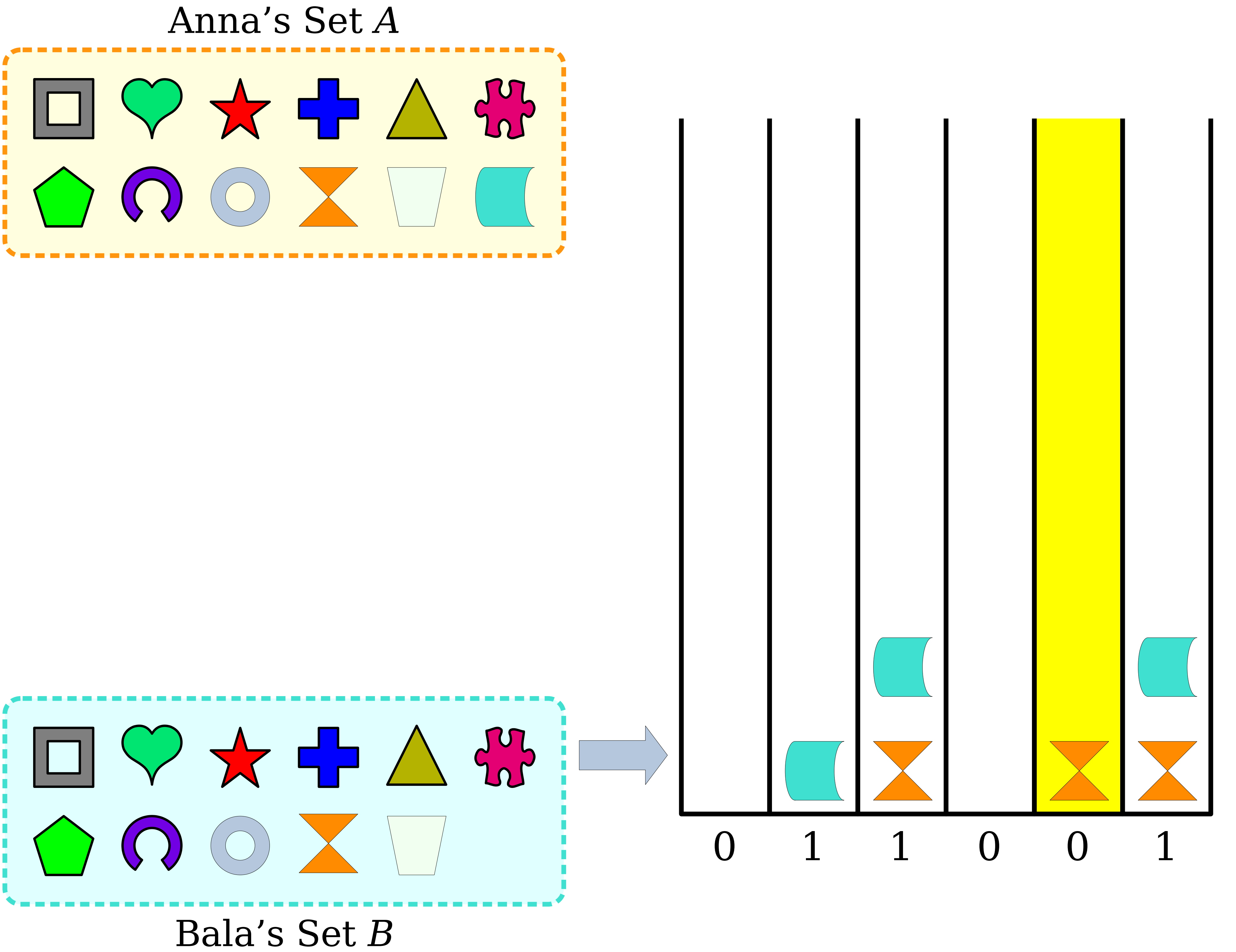
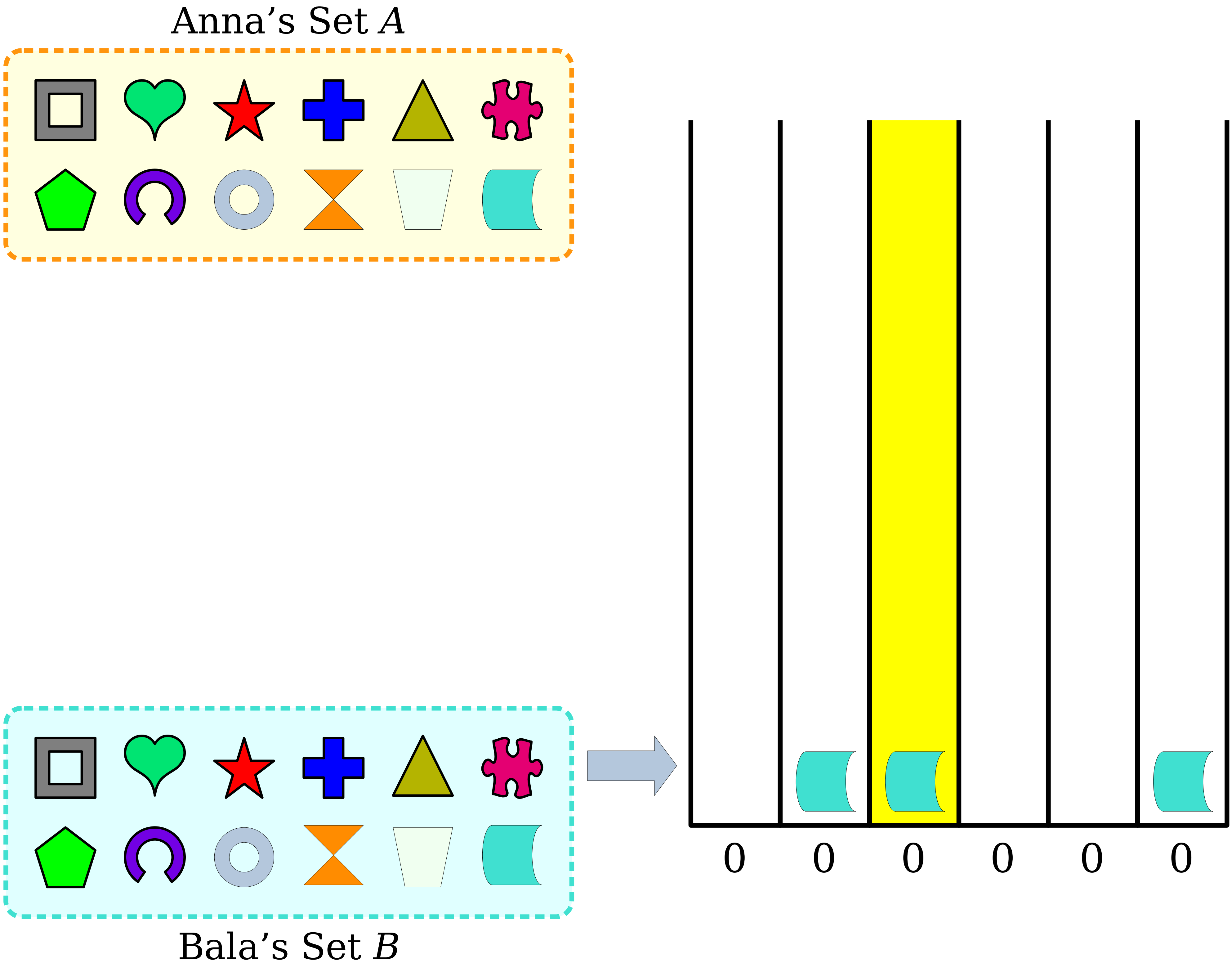
1
2



Anna's Set A

Bala's Set B


Anna's Set A
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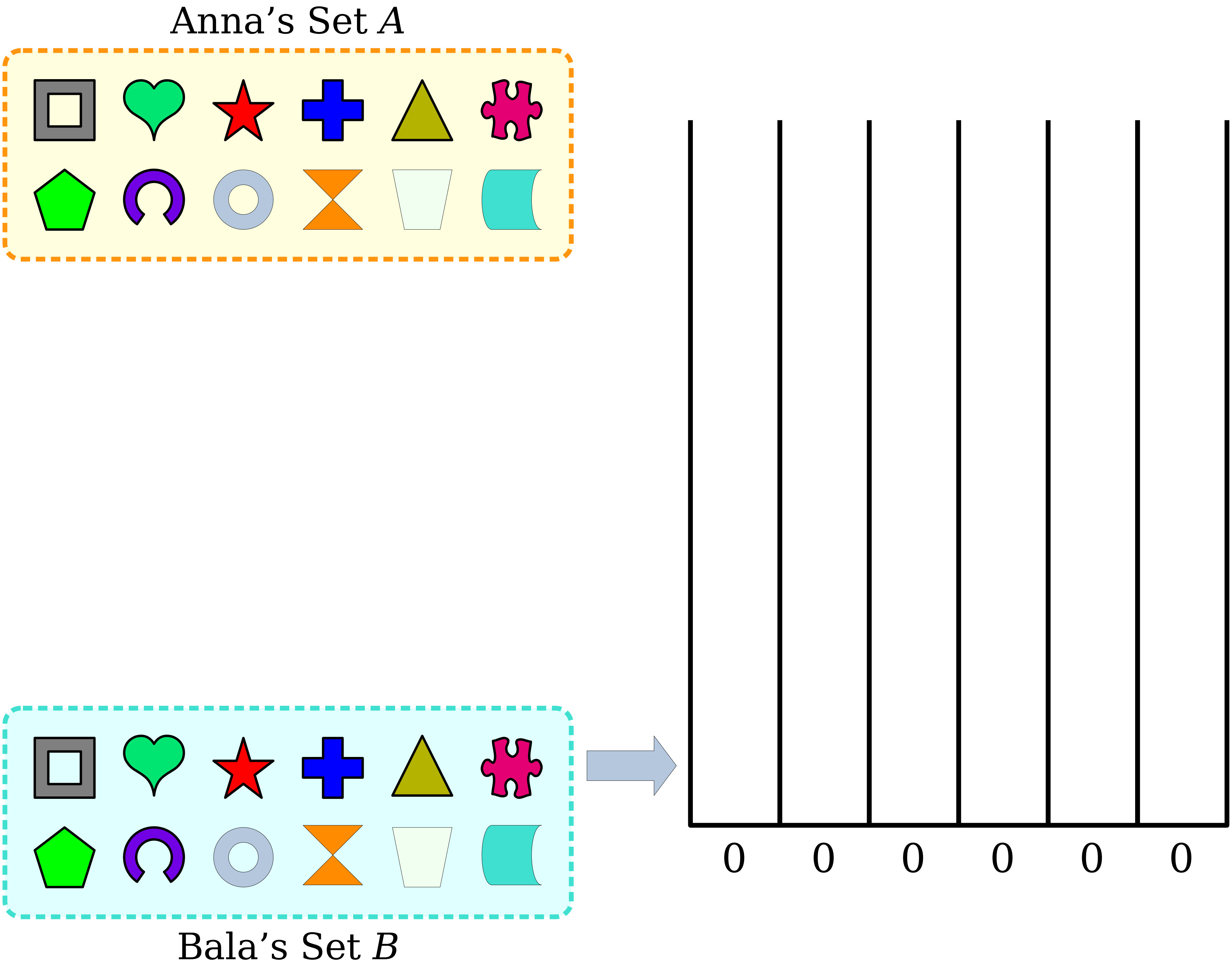
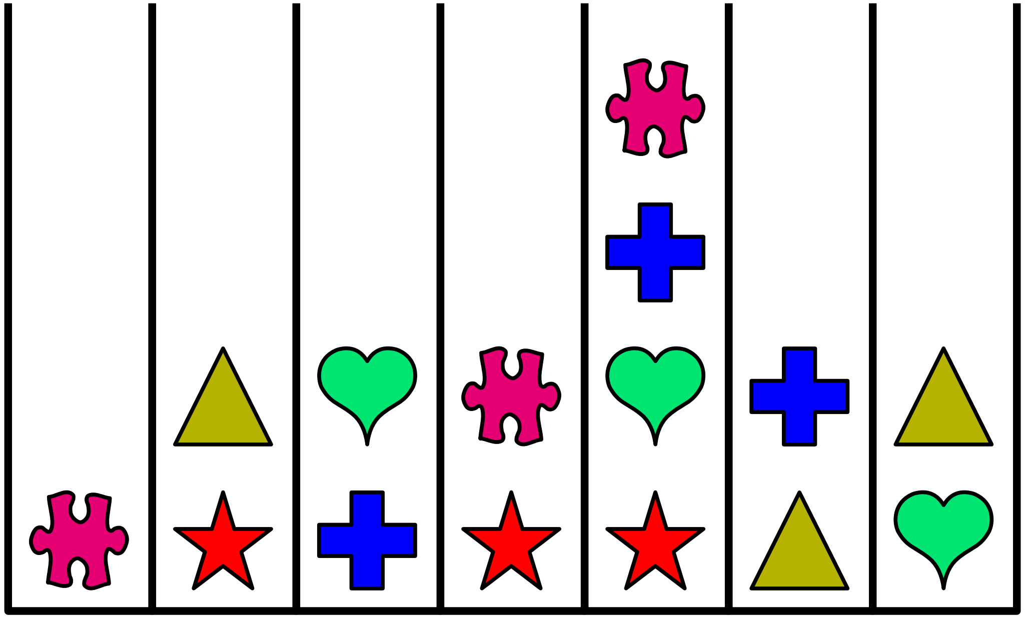
Bala's Set B


Anna’s Set A
Bala’s Set B
0
0
0
0
0
0

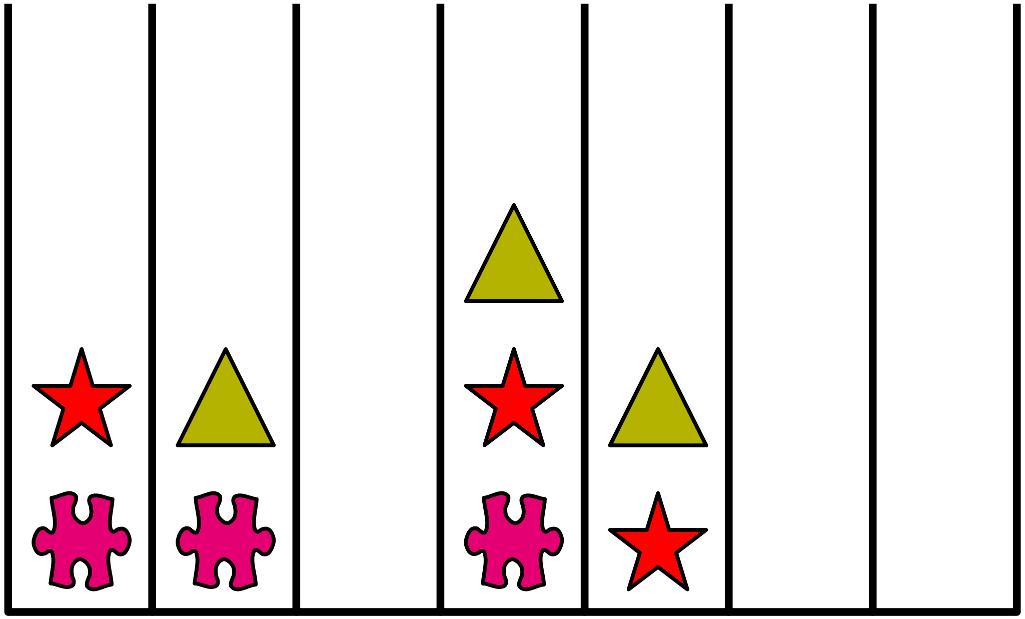
The Peeling Algorithm

- If we have multiple copies of each item, we might still be able to recover them all even when collisions exist.
- ***Algorithm:*** Repeatedly find a bucket with one item in it, identify the item, and remove all copies of it.
- (To perform the last step: hash the item with all the hash functions, XOR it out of the buckets it's in, and decrement the appropriate counters.)



The Peeling Algorithm

- Peeling doesn't always work; we can get "stuck" with no more peelable elements.
- ***Question:*** How likely is it that we'll be able to peel all the elements away?



The Peeling Algorithm

- We have three parameters to consider:
    - k, the number of items to distribute.
    - m, the number of buckets we choose.
    - d, the number of hash functions we pick. (Equivalently, the number of copies of each item we distribute across the buckets.)
- We can't control k. However, we can pick m and d.
- We want to know the chance that we can recover ***every*** element through peeling.
- What happens to the peeling probability as we vary m?
- What happens to the peeling probability as we vary d?
- Think about what happens if m and d are each quite small or quite large.

Answer at
https://cs166.stanford.edu/pollev

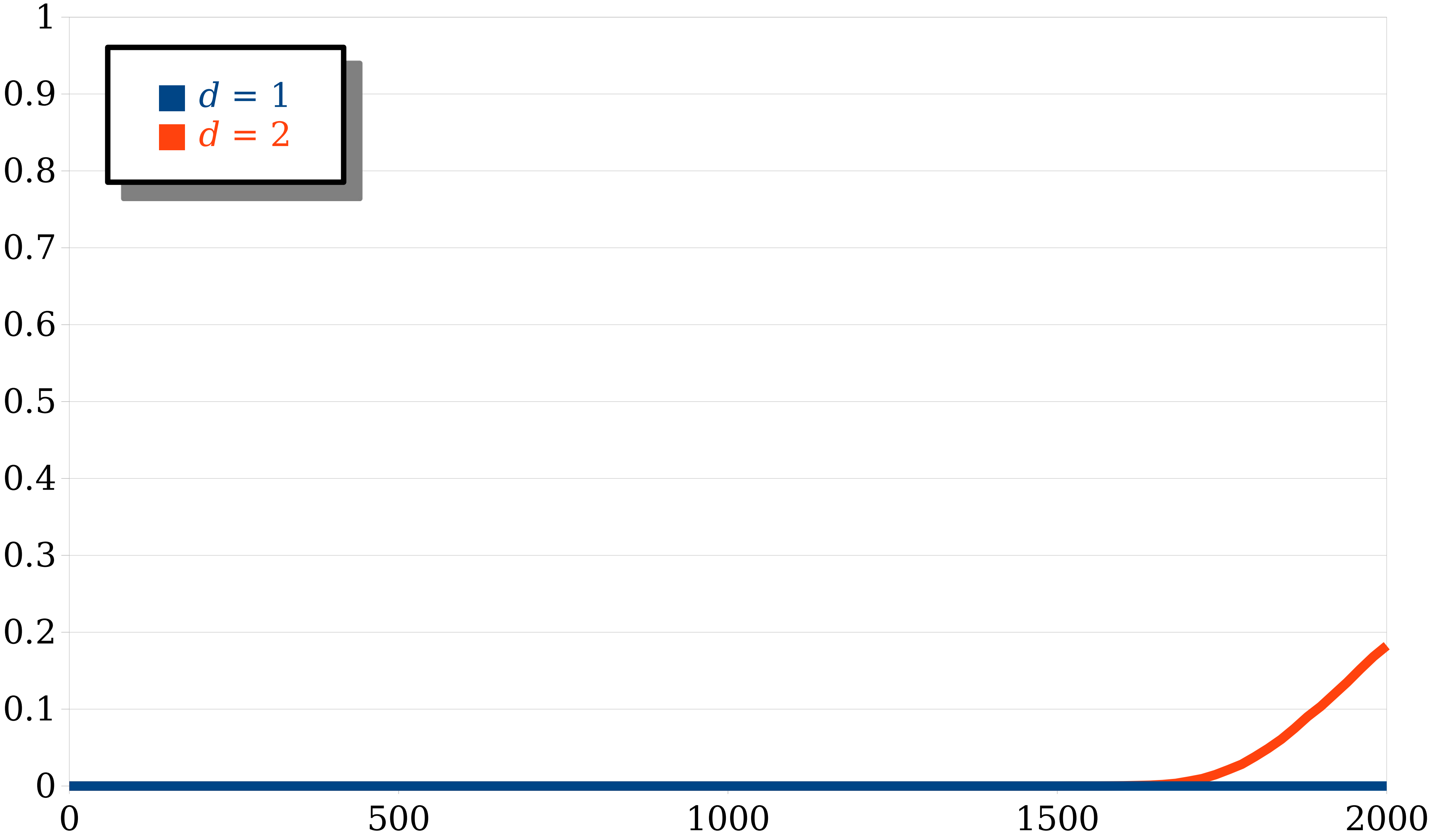
Tuning *m*

- If *m* is too small, we expect the success probability to drop to zero.
  - Extreme case: All in one bucket means we can never peel.
- If *m* gets larger, we expect the success probability to increase toward one.
  - Extreme case: with infinitely many buckets, we never get collisions.
- ***Goal:*** Make *m* large enough to ensure good success probability, but small enough that we don’t have to transmit too much data.

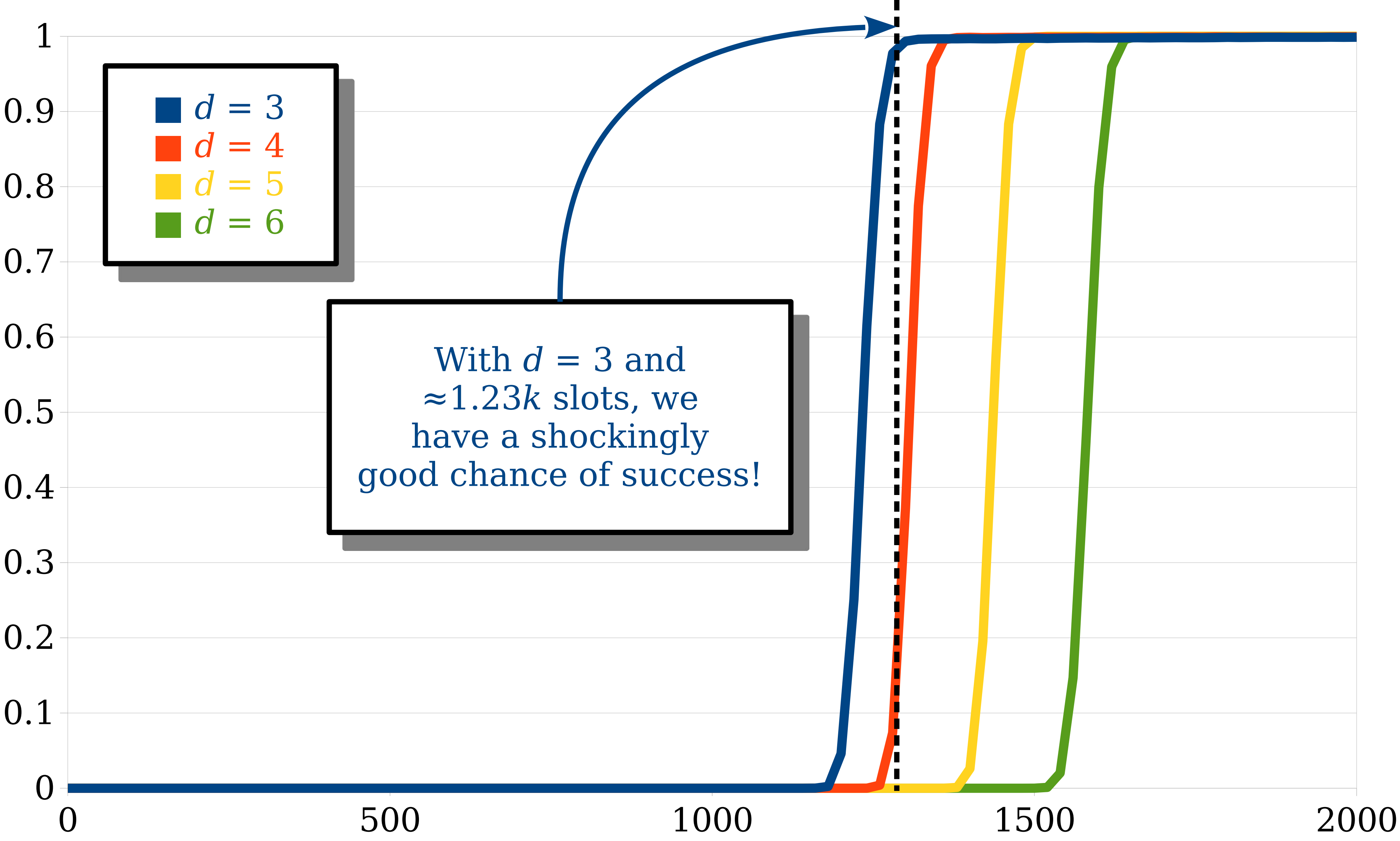
Tuning *d*

- The impact of d is a bit more subtle.
- If d is too small, collisions will be harder to resolve.
  - Extreme case: if $d = 1$, we're back in the same situation we started with earlier.
- If d is too large, we'll get so many collisions we won't be able to find anything to peel.
  - Extreme case: if $d = m$, peeling always fails.
- ***Goal:*** Find a "sweet spot" for d to ensure the highest probability of success.

Pro Tip: When designing a data structure, it never hurts to get empirical data first!



What is the probability that all elements are peeled when we have $k = 1000$ elements and m buckets, as a function of d?

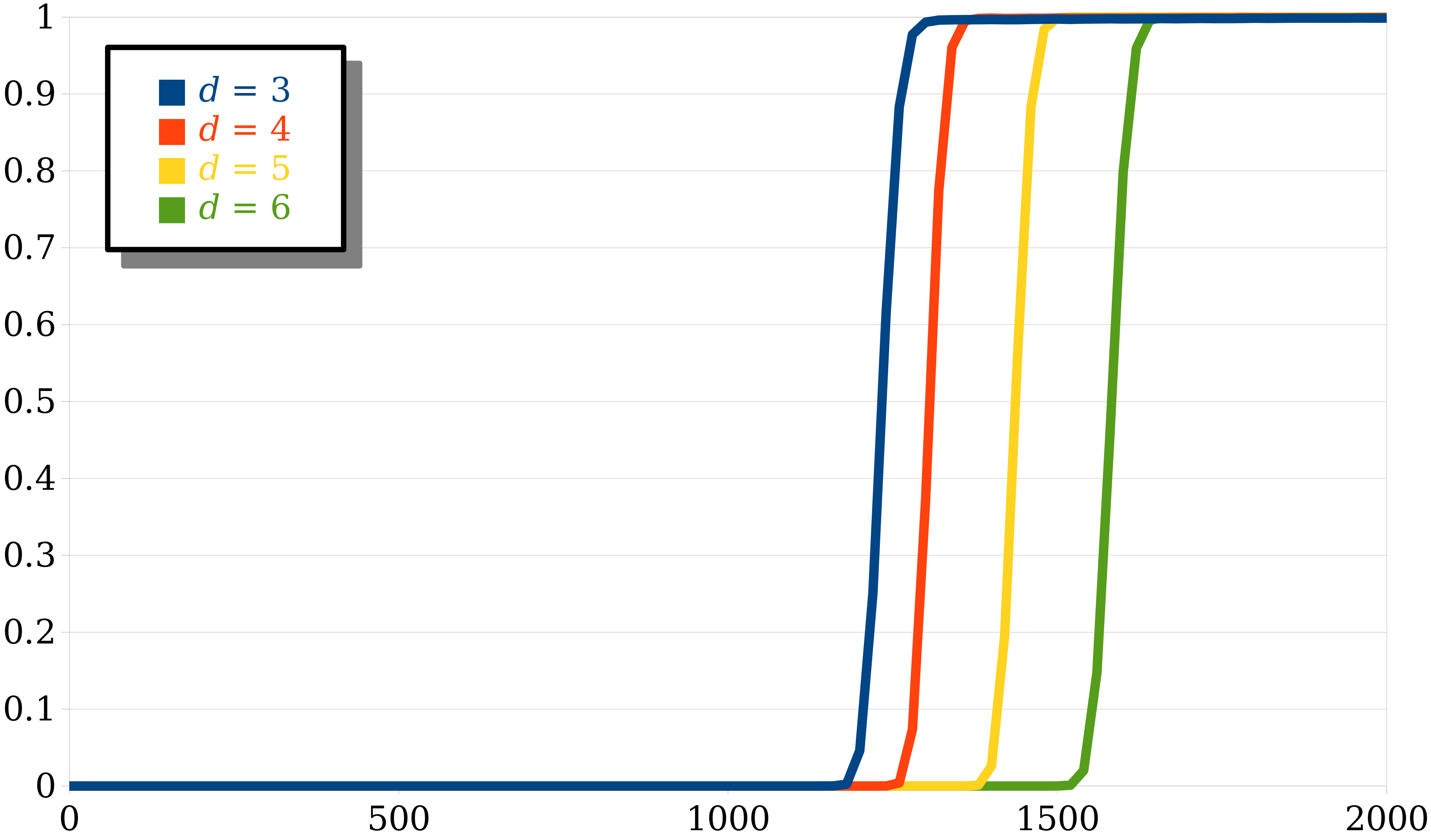


With $d = 3$ and $\approx 1.23k$ slots, we have a shockingly good chance of success!

What is the probability that all elements are peeled when we have $k = 1000$ elements and m buckets, as a function of d?

IBLTs

- Create an array of ≈1.23k empty buckets. Choose 3 hash functions h_1, h_2, and h_3. Assume they behave truly randomly.
- We support three operations:
  - ***add***(x): Add x to buckets $h_1(x)$, $h_2(x)$, and $h_3(x)$.
  - ***remove***(x): Remove x from buckets $h_1(x)$, $h_2(x)$, and $h_3(x)$.
  - ***list***(): Run the peeling algorithm.
- Anna ***add***s her items, Bala ***remove***s his, and then either can ***list*** the items to find what's missing.
- This is called an ***IBLT*** (***I***nvertible ***B***loom ***L***ookup ***T***able).
- It requires **≈1.23*k*** buckets to be transmitted, has success probability **1 – O(k^{-1})**, and requires very little computation by the two parties.
  - Or you can use four hash functions, ≈1.31k buckets, and have success probability 1 – O(k^{-2}); or five hash functions, ≈1.44k buckets, and have success probability 1 – O(k^{-3}), etc.



What’s going on here?

Goal: Explain why the phase transition exists for $d \geq 3$.

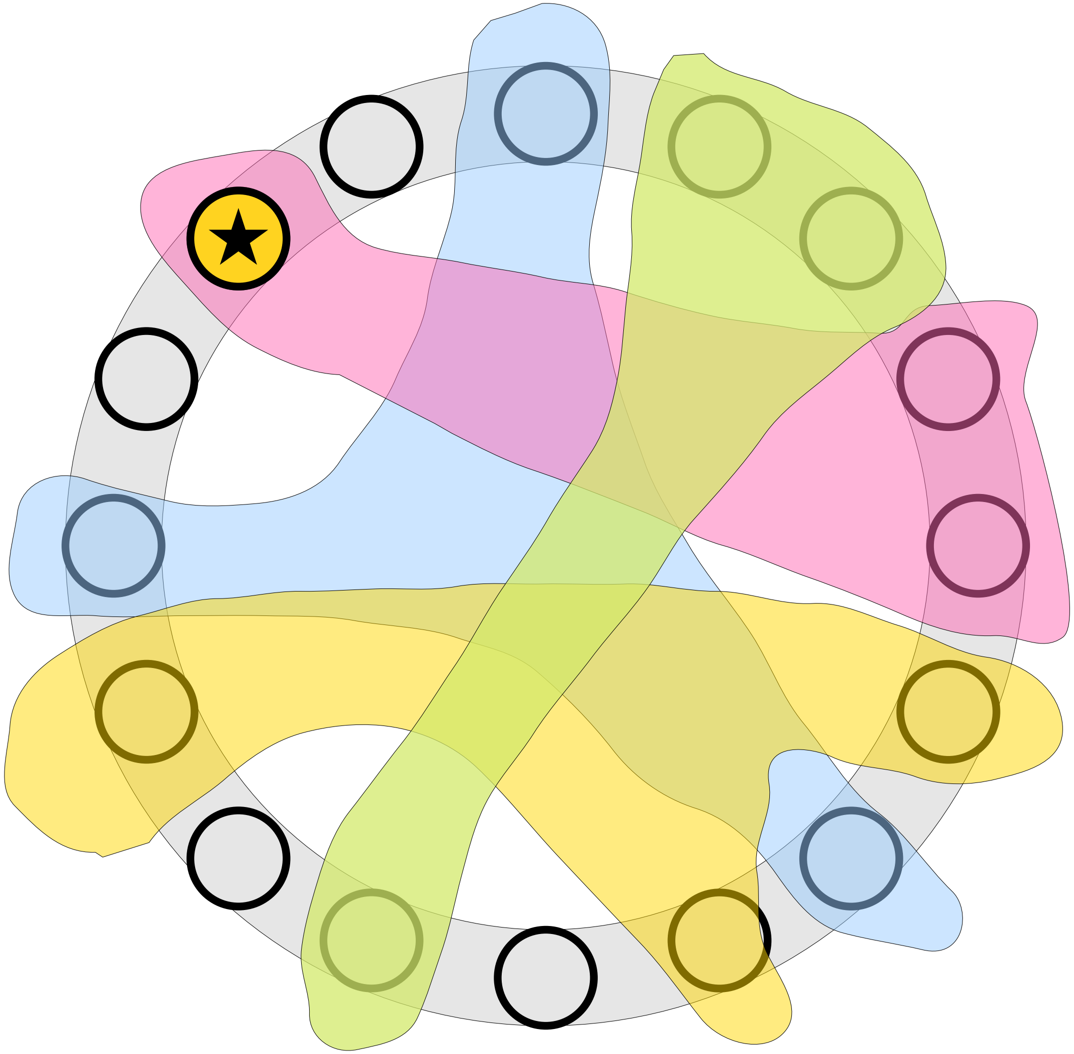
Where We’re Going

- ***Step One:*** Model the peeling process as an operation on hypergraphs.
- ***Step Two:*** Determine the probability that peeling succeeds on a random hypergraph.
- ***Step Three:*** Analyze our results from Step Two to see where the phase transition comes from.

Step One: Model peeling as a process on random hypergraphs.

Modeling Peeling

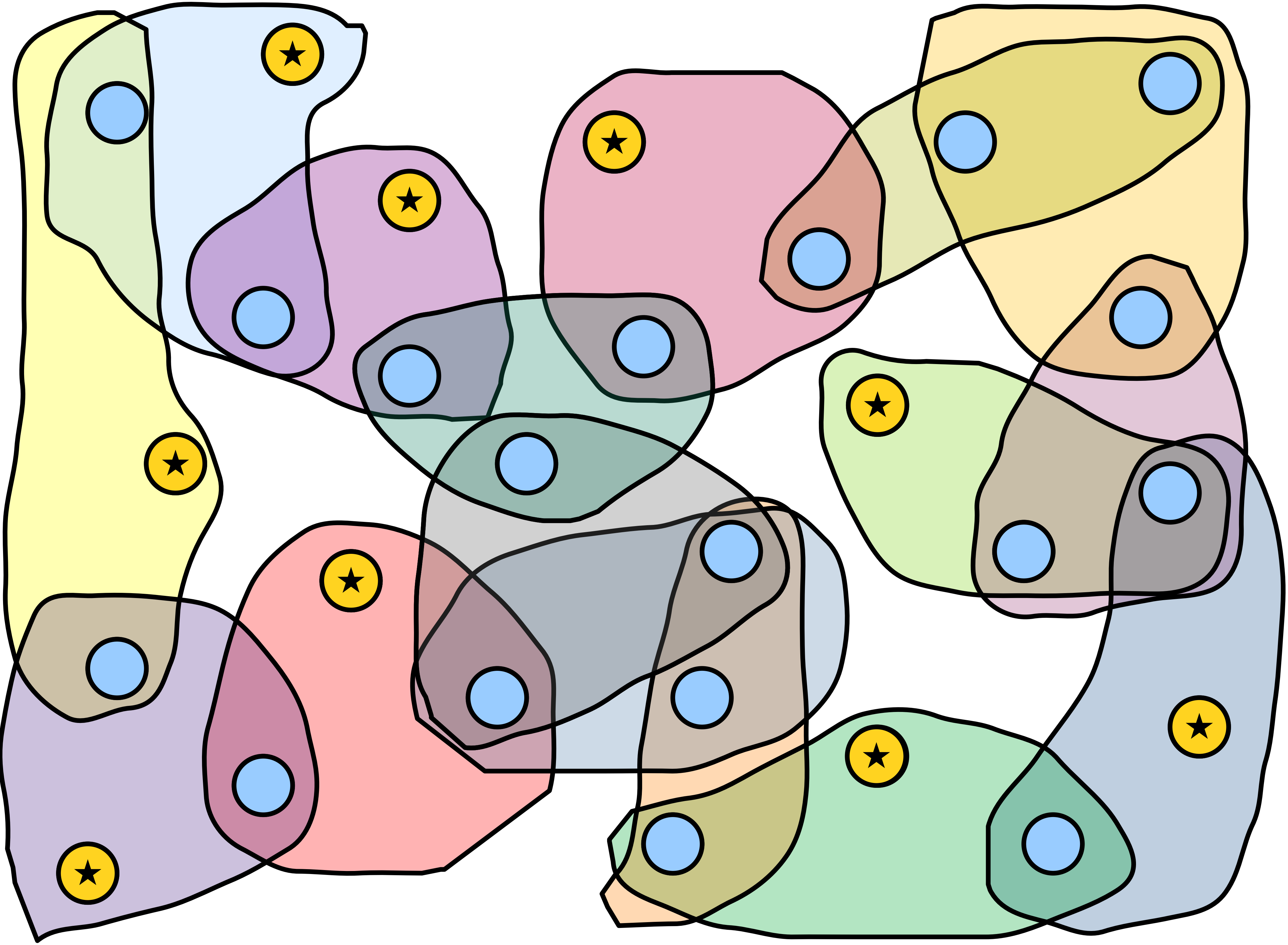
- Model the assignment of items to buckets as a hypergraph.
    - Each bucket is a node.
    - Each item is a hyperedge linking *d* buckets together.
- Model peeling as finding a node with degree one or less and deleting it and any edges touching it.

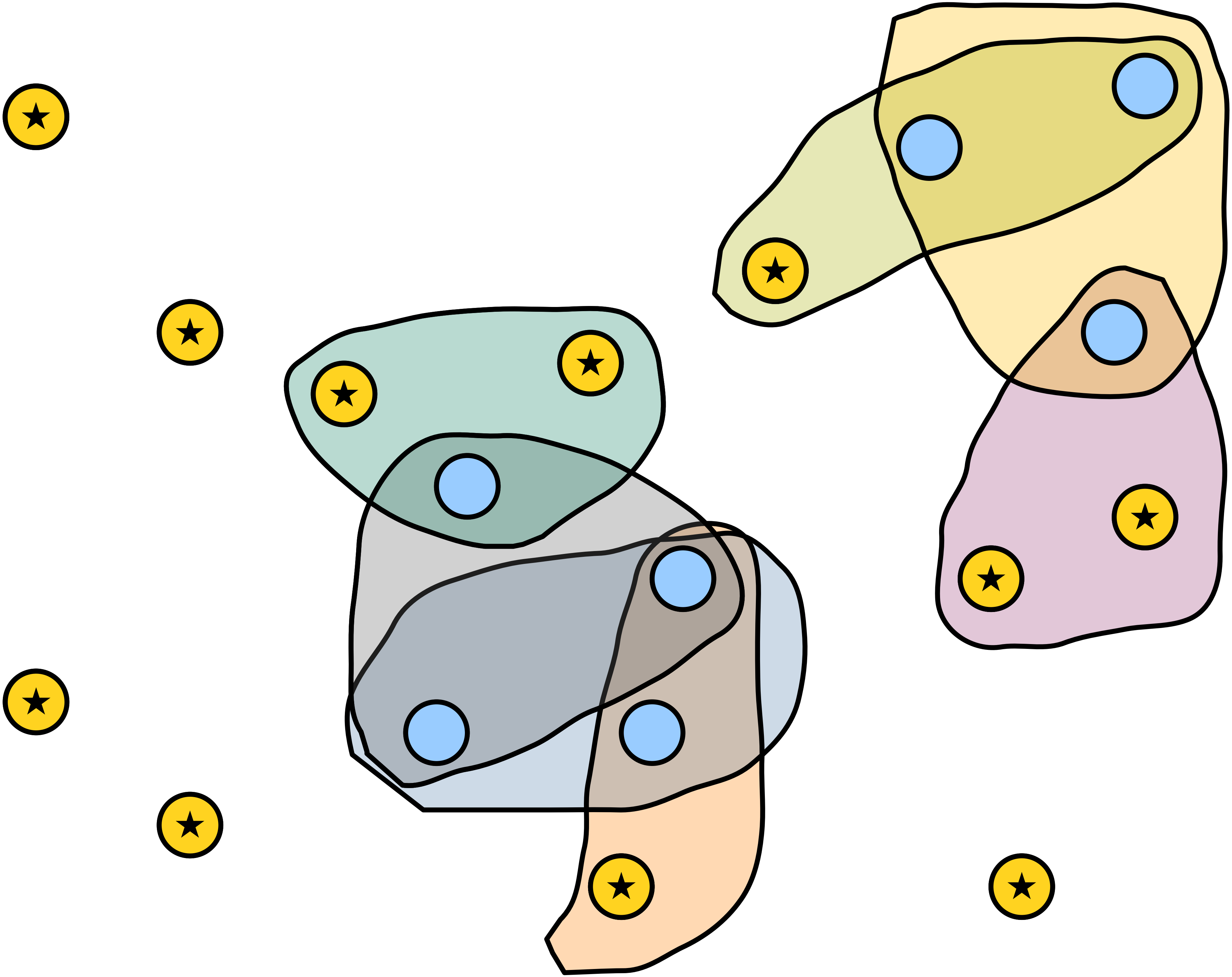


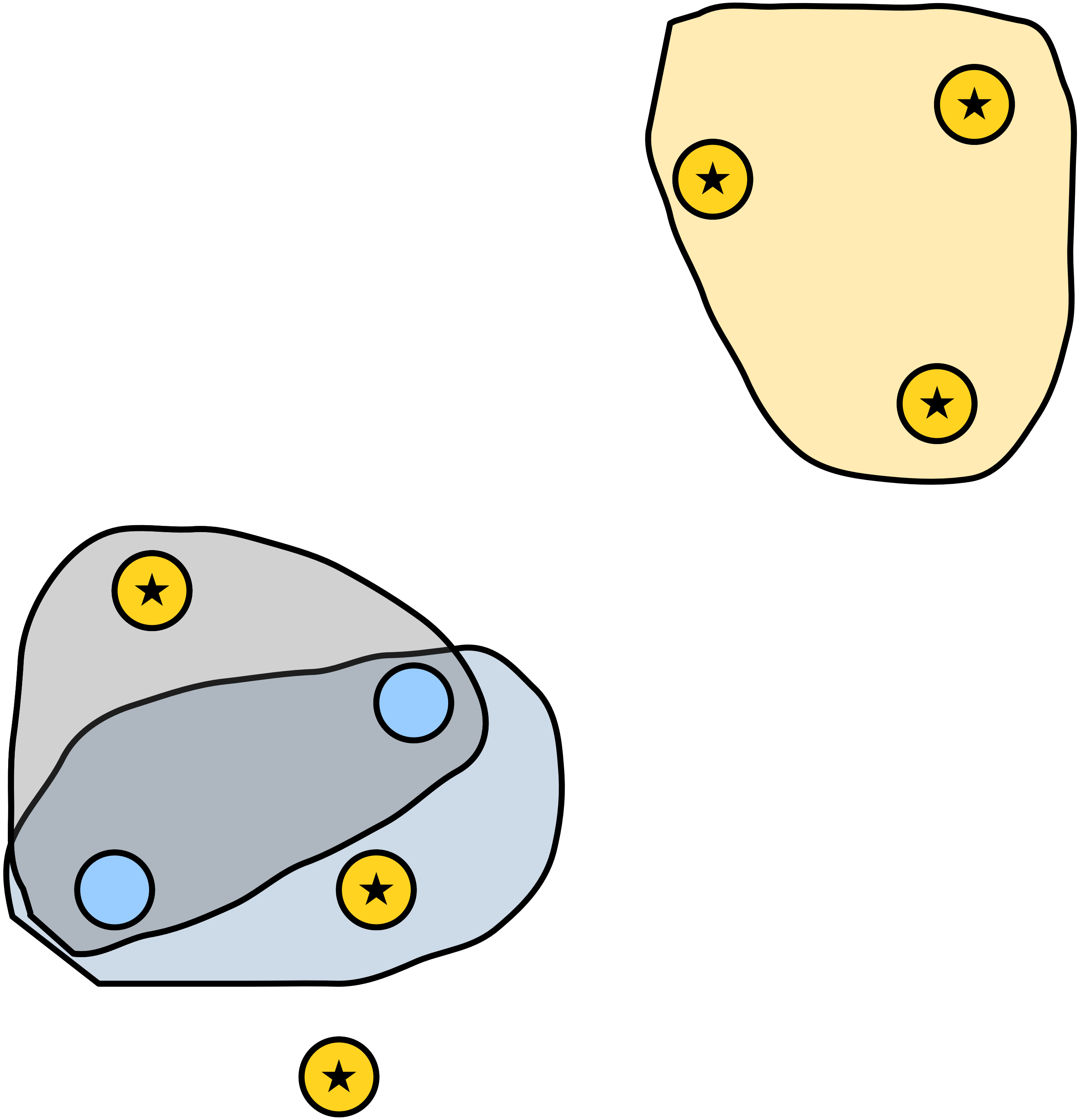
Question: Given a random hypergraph with V nodes and $E = \alpha V$ hyperedges, each of which links d nodes, what's the probability that every node is peeled?

Claim: It’s easier to analyze a parallel version of this process.

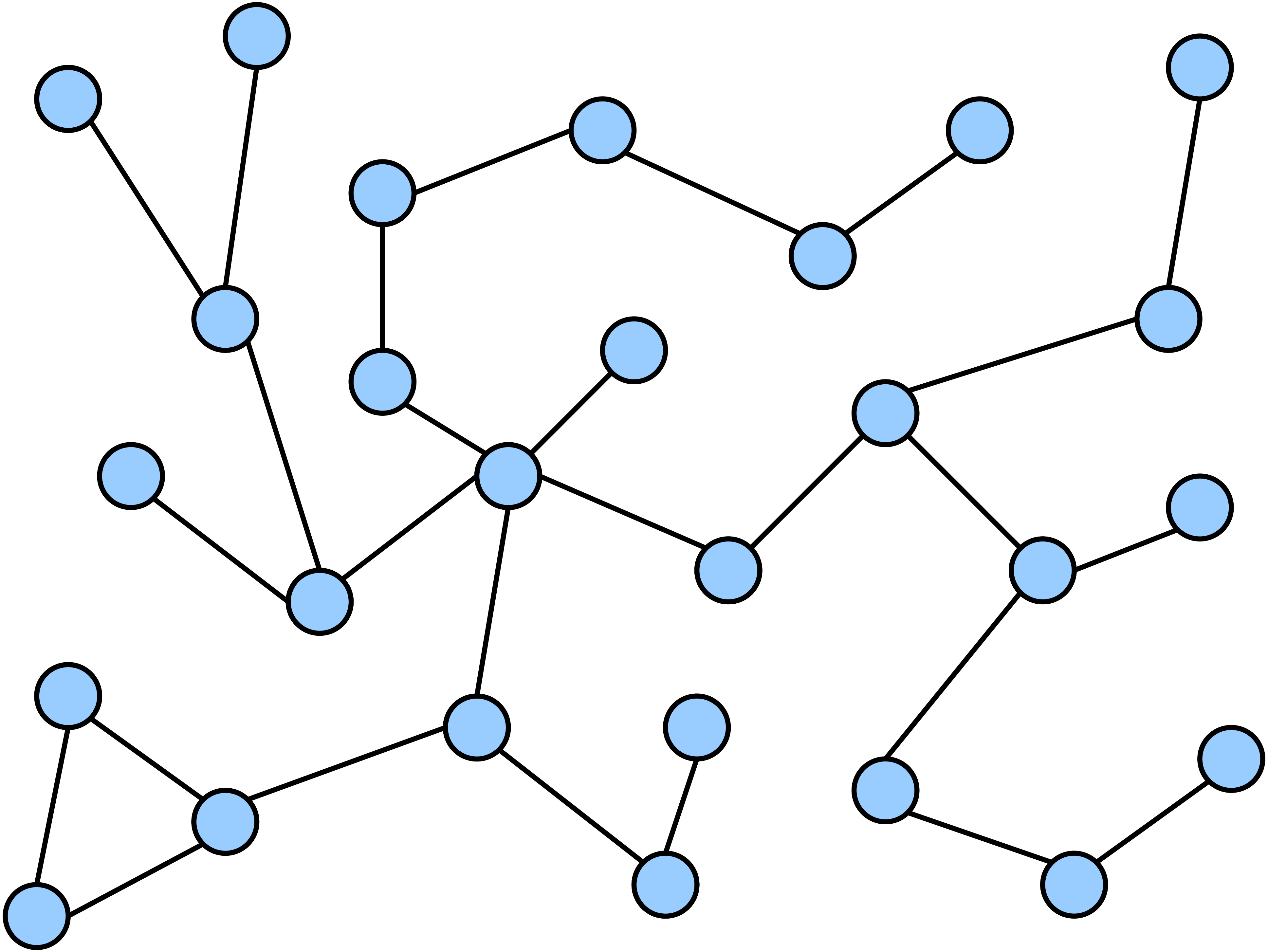
One “round” consists of identifying and peeling all nodes of degree 0 or 1.

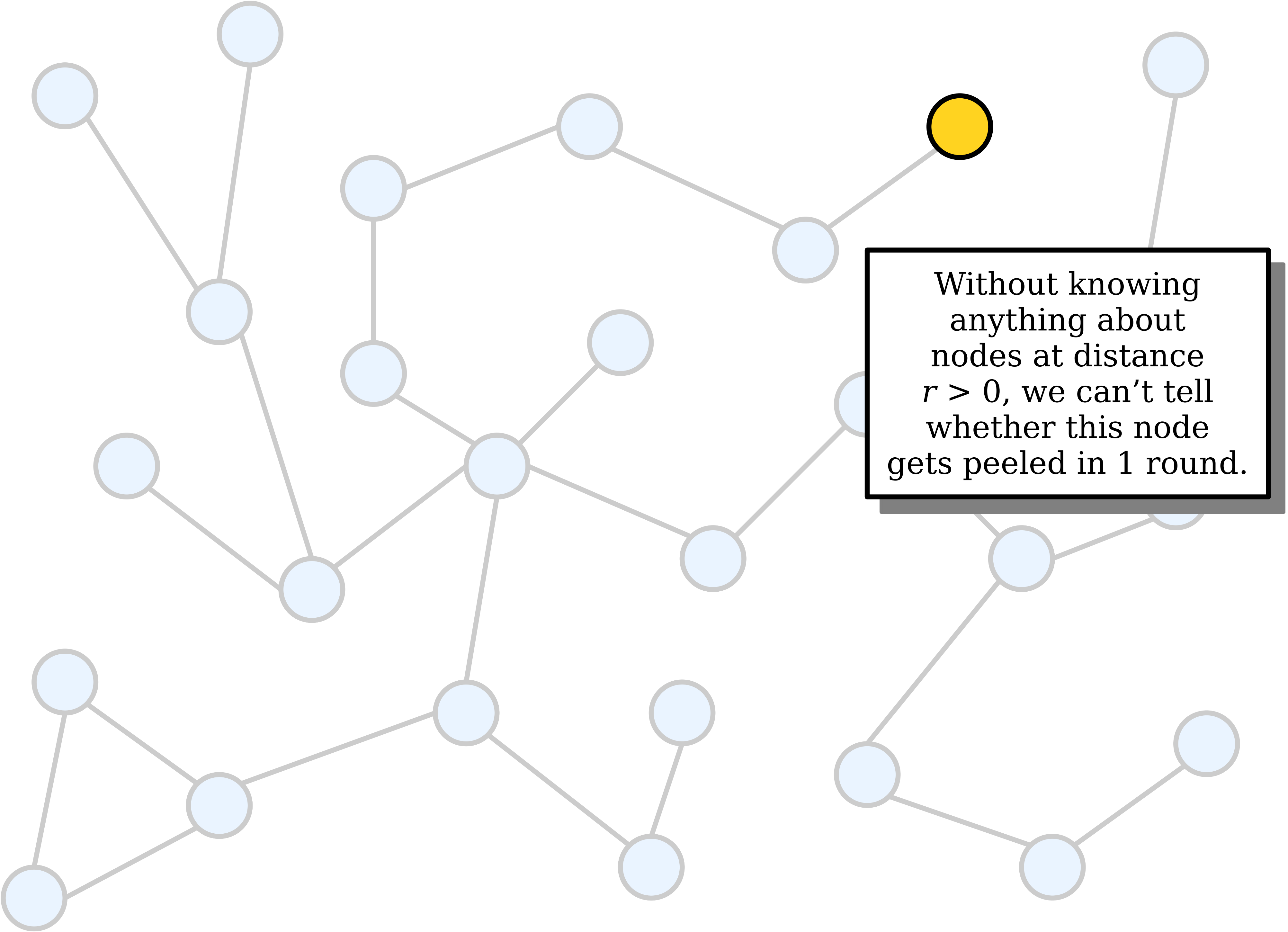




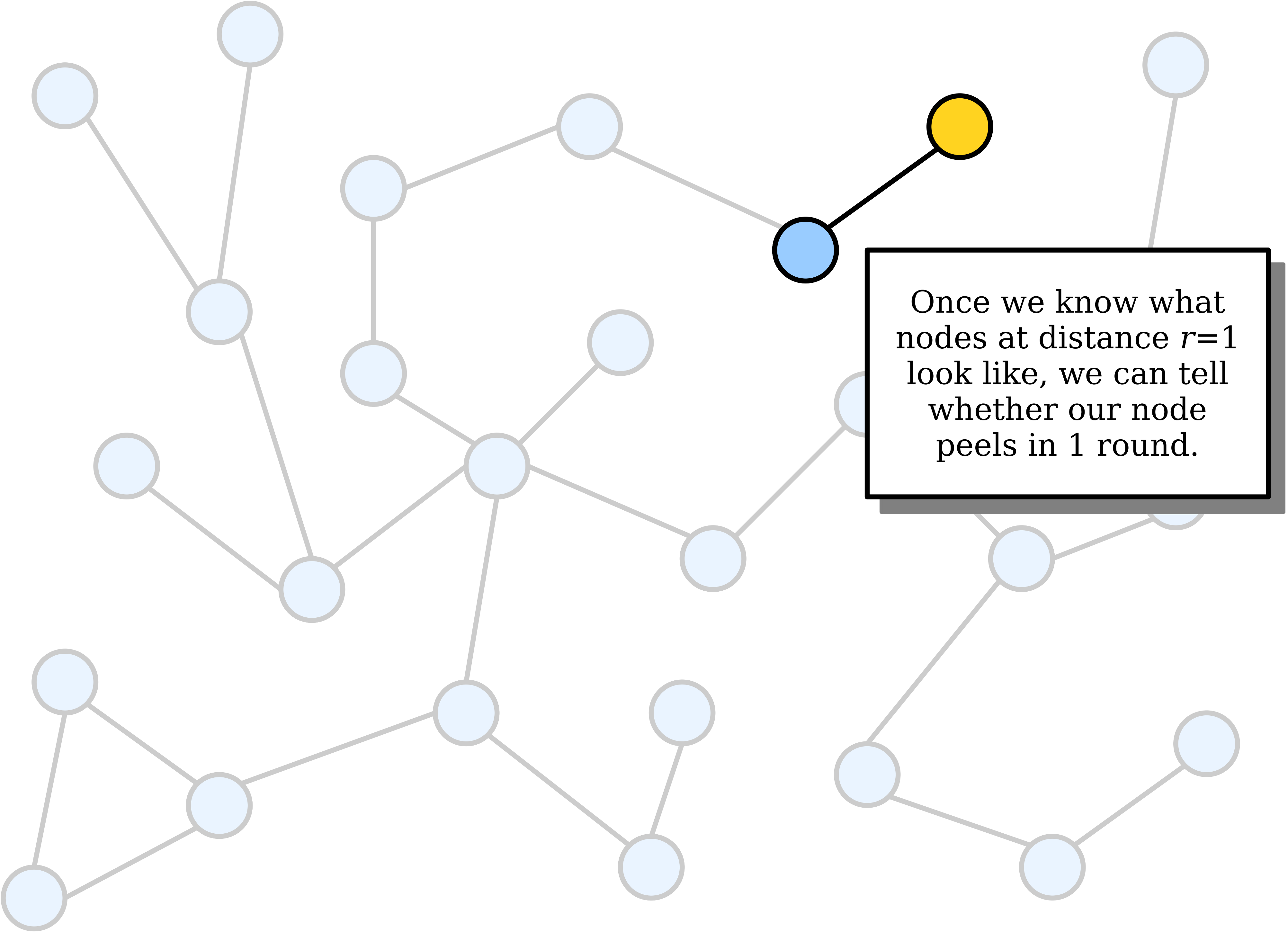


Goal: Determine the probability that a node $v \in V$ is peeled after r rounds.

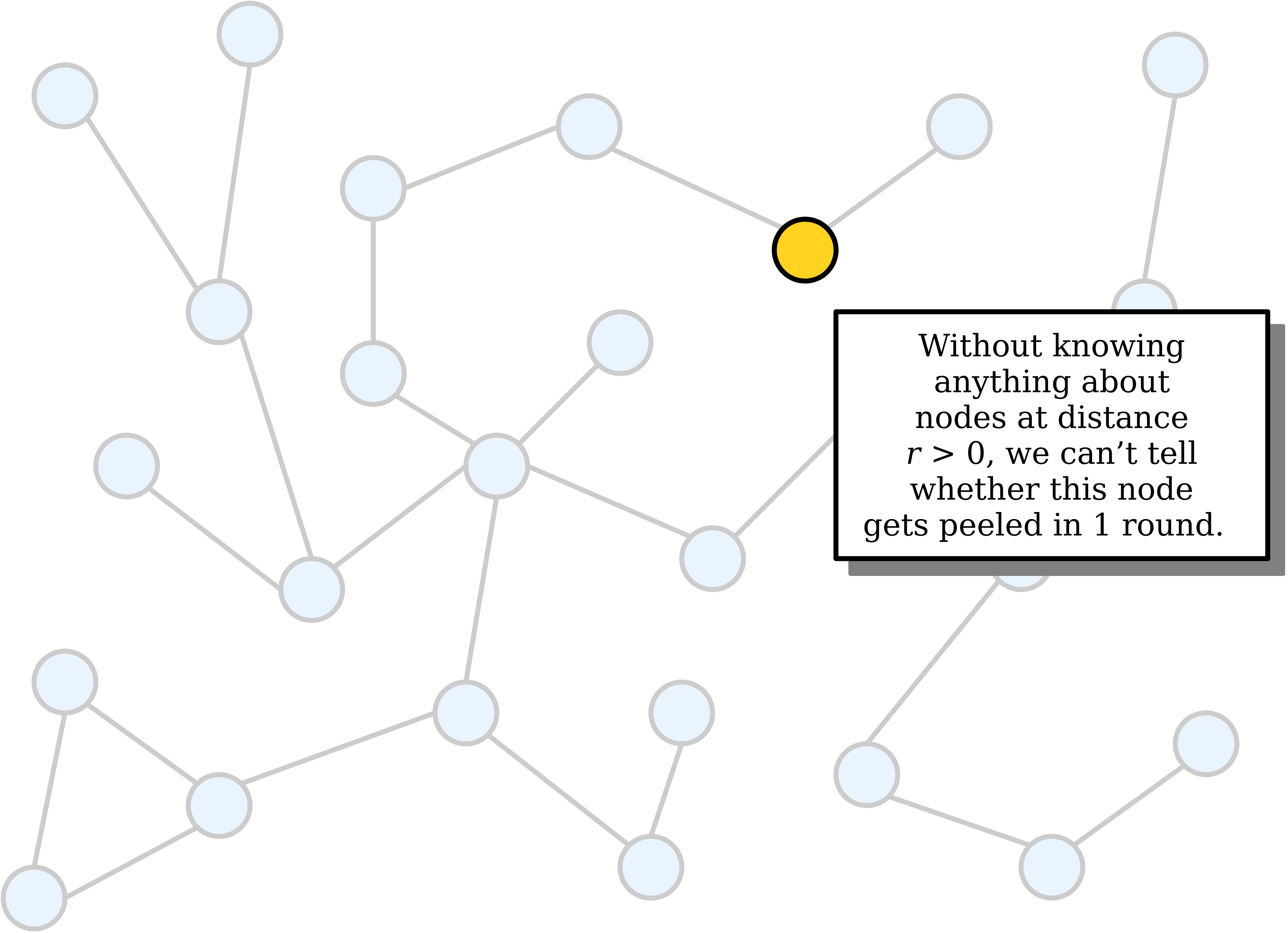




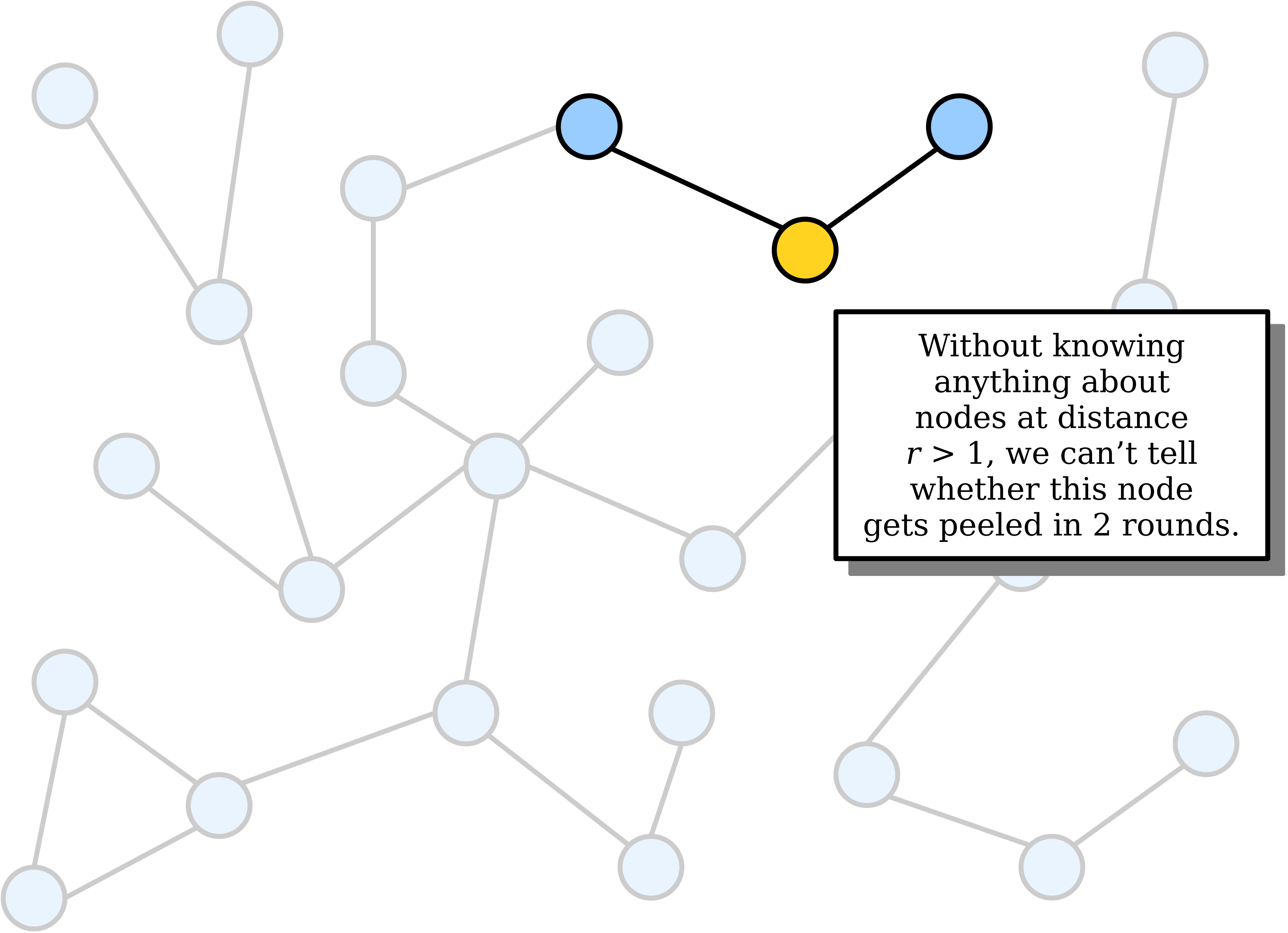
Without knowing anything about nodes at distance $r > 0$, we can’t tell whether this node gets peeled in 1 round.



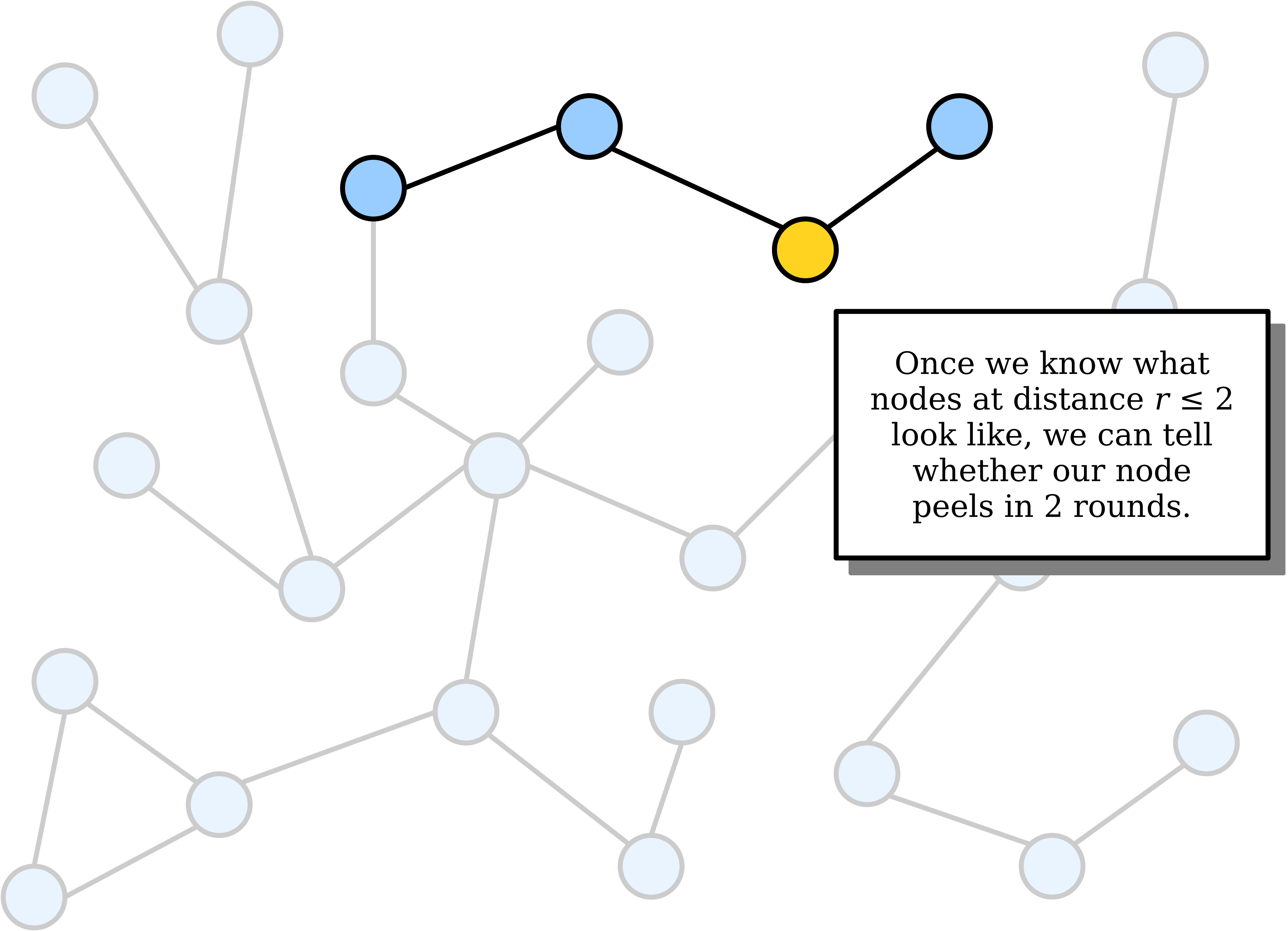
Once we know what nodes at distance r=1 look like, we can tell whether our node peels in 1 round.



Without knowing anything about nodes at distance $r > 0$, we can’t tell whether this node gets peeled in 1 round.



Without knowing anything about nodes at distance $r > 1$, we can’t tell whether this node gets peeled in 2 rounds.



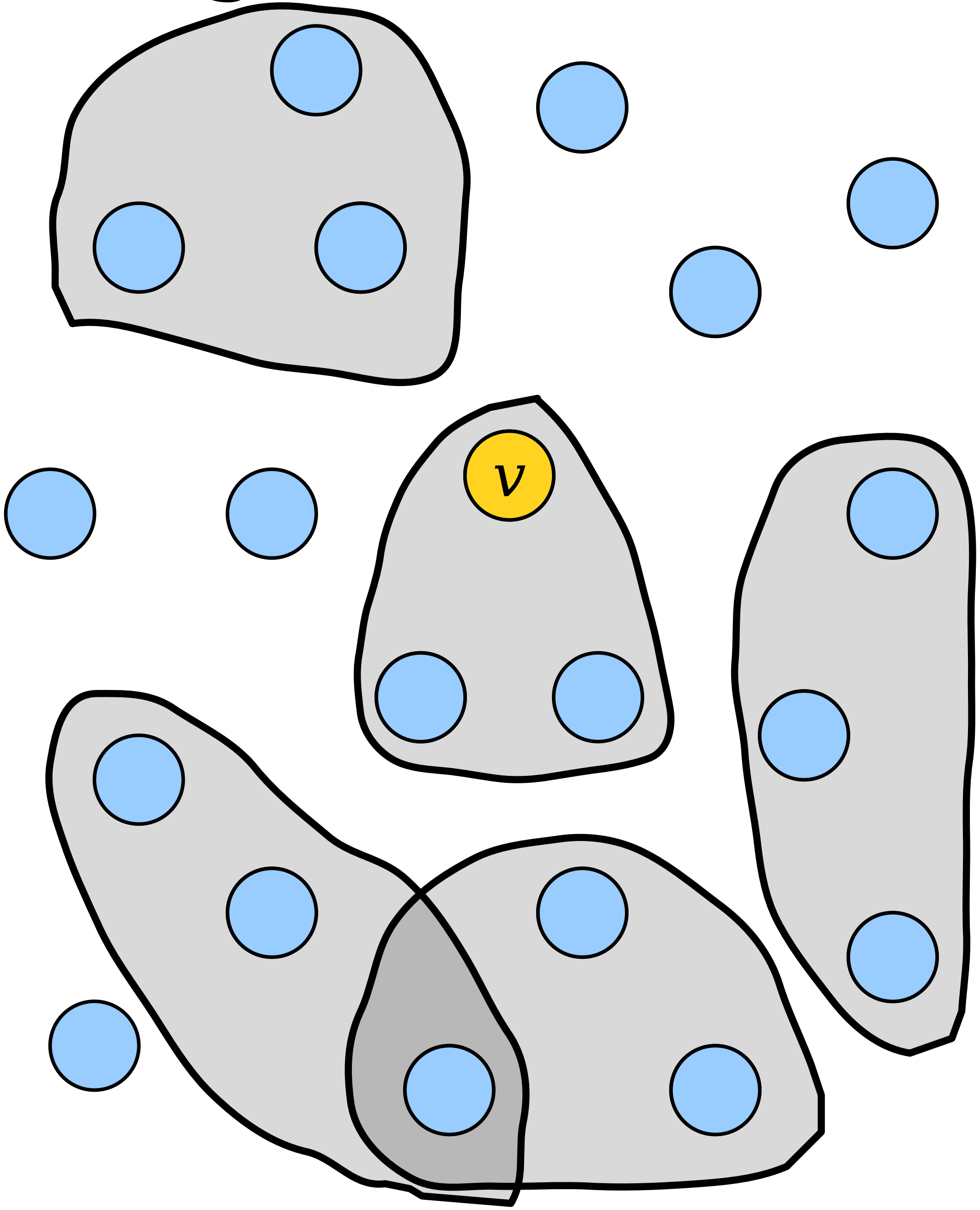
Once we know what nodes at distance $r \leq 2$ look like, we can tell whether our node peels in 2 rounds.

Observation: To determine whether v is peeled within r rounds, we only need to focus on nodes and edges at distance r or less from v.

Algorithms Question: How would you find all the nodes at distance r or less from a starting node?

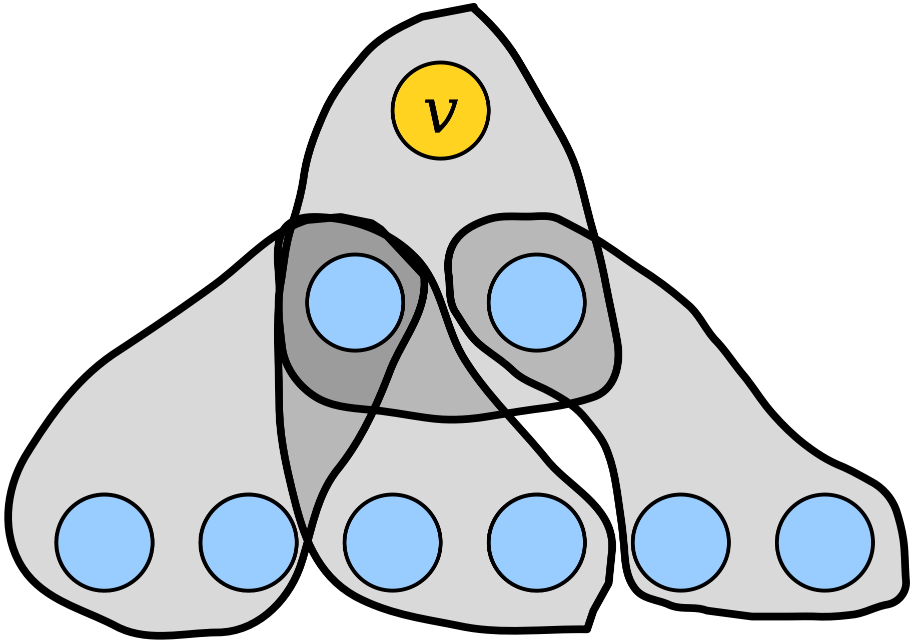
Exploring the Neighborhood

- Pick any node $v \in V$.
- Focus on any one hyperedge in the graph. What is the probability that it includes v?
  - ***Answer:*** $^{d}/_{V}$.
- There are $E = \alpha V$ hyperedges, so the degree of v is a Binom(αV, $^{d}/_{V}$) variable.
- We'll approximate this as a **Poisson(αd)** variable.



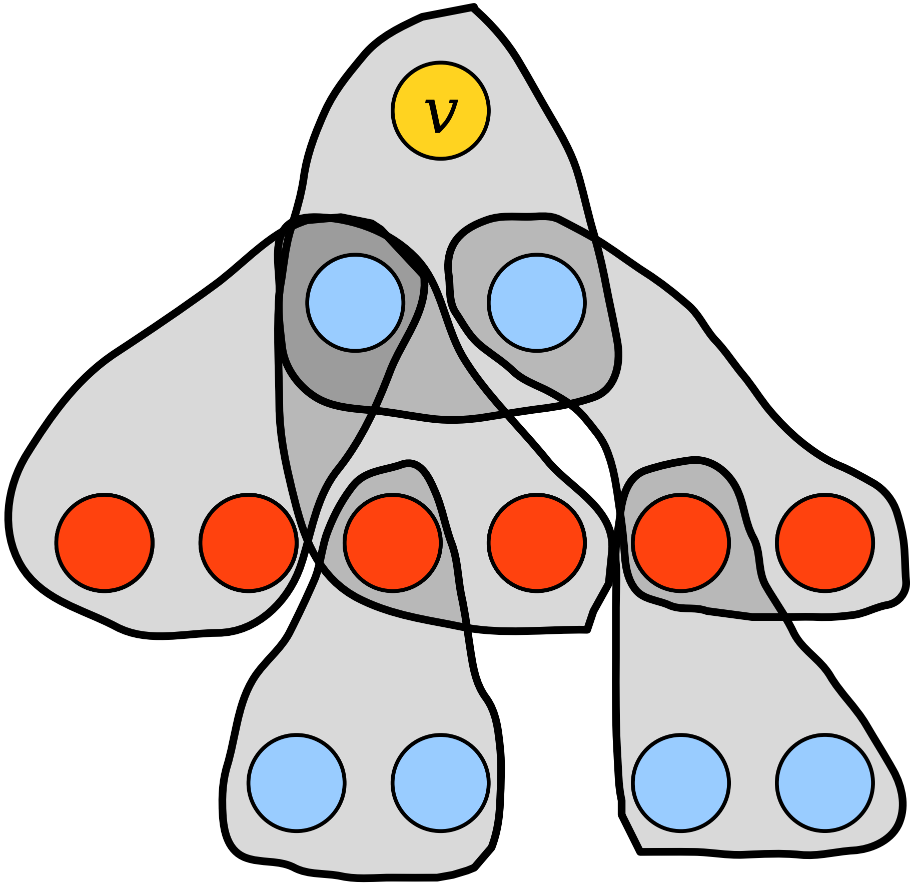
Exploring the Neighborhood

- Look at the nodes one hop from *v*.
- Think about where their outgoing edges go.
- ***Claim:*** As *V* tends to infinity, the likelihood that these edges include a cycle drops to zero. *(Why?)*
- Thus their outgoing edges lead away from these nodes along the lines of a tree.



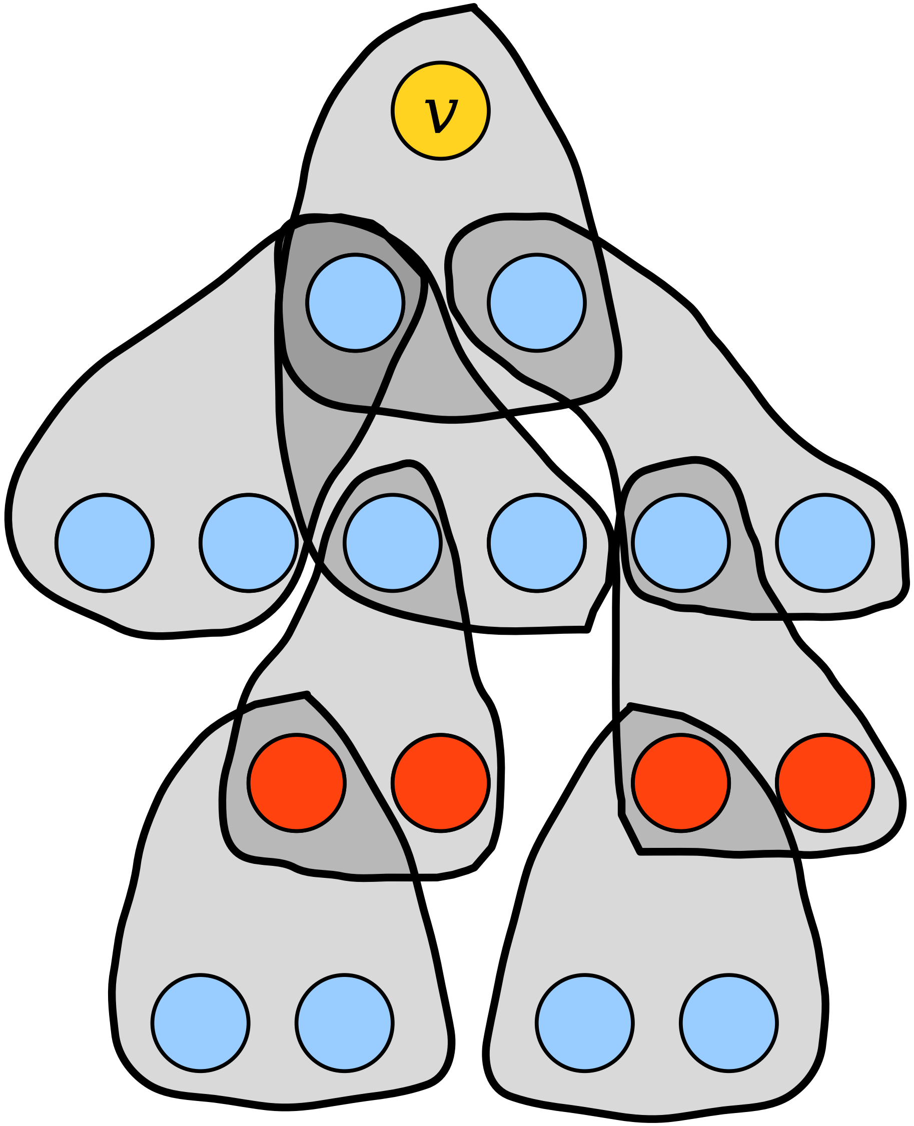
Exploring the Neighborhood

- Now look at the nodes two levels down.
- Excluding the edge to *v*, what's the distribution of the degrees of these nodes?
- ***Claim:*** As *V* approaches infinity, each node's degree is well-described as a **Poisson(αd)** variable.
  - *(What details are we ignoring? Why can we ignore them?)*
- Additionally, the chance that these edges close a cycle drops to zero as *n* goes to infinity.



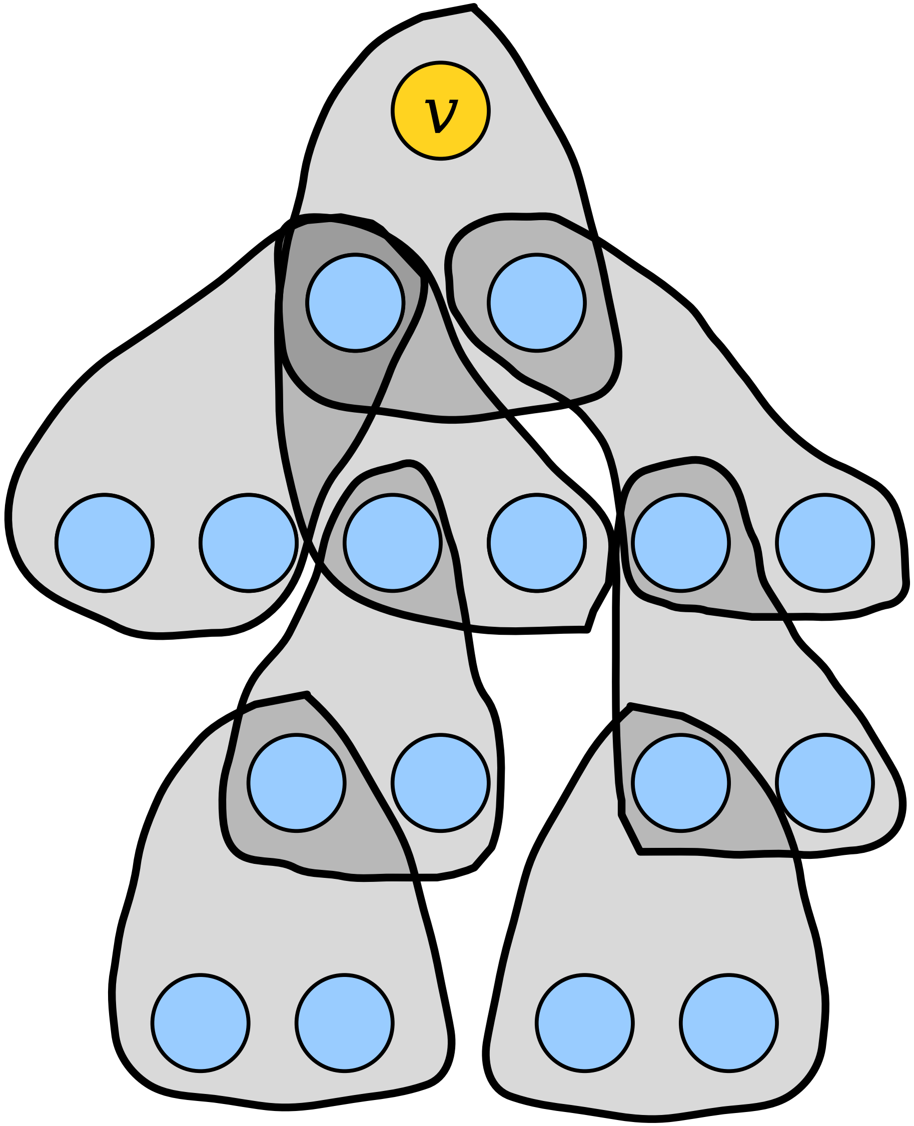
Exploring the Neighborhood

- Now look at these new nodes.
- ***Claim:*** As before, for sufficiently large *V*, the probability any outgoing edges from these nodes cause a cycle drops to zero.
- And these nodes' degrees, excluding their parent edges, are well-modeled as **Poisson(*αd*)** variables.



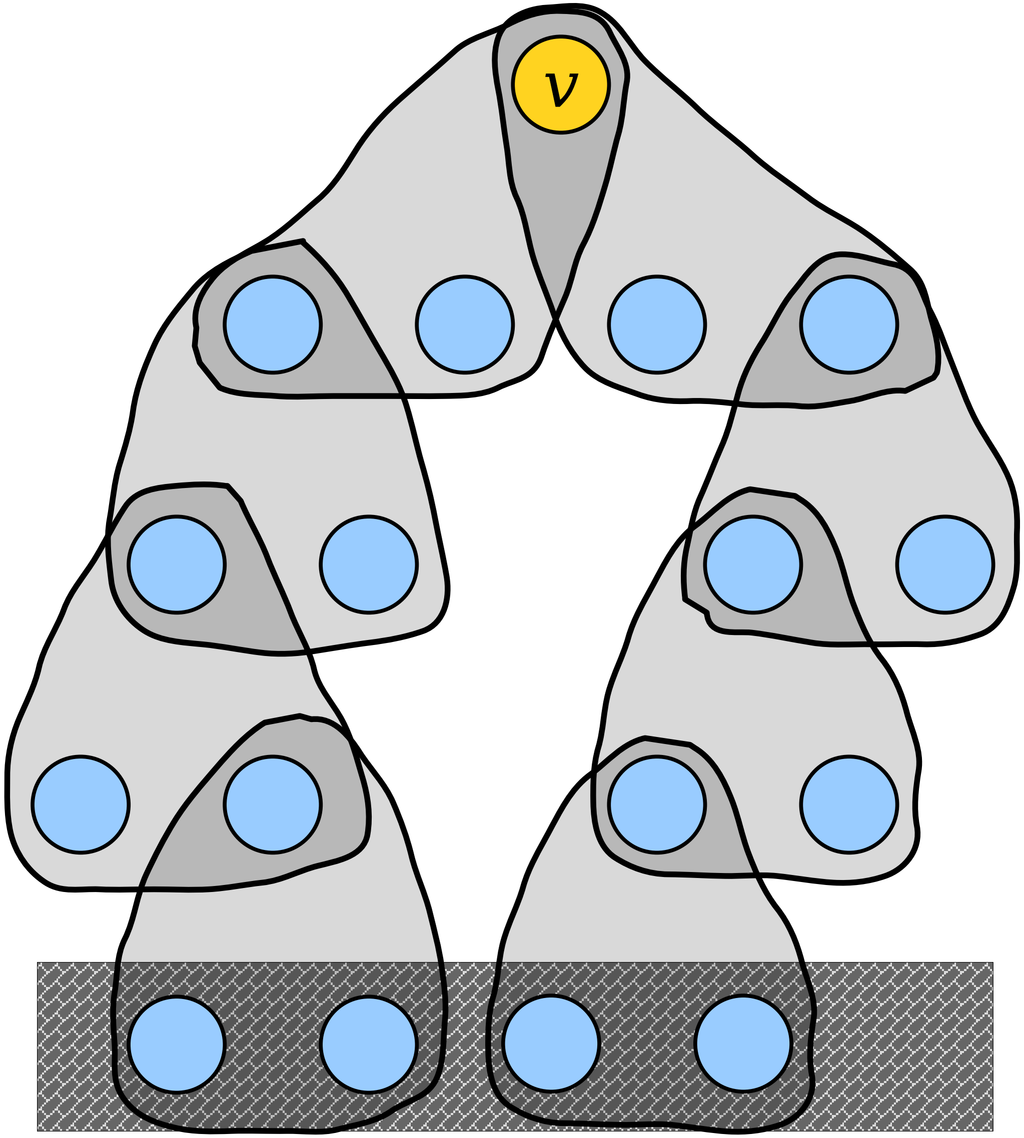
Exploring the Neighborhood

- ***General Claim:*** For any fixed constant r, and as V tends toward infinity:
  - The neighborhood of radius r around v is tree-shaped.
  - Each node's child edge count is distributed as Poisson(αd) variables.
- ***Question:*** For a fixed constant r, what is the probability that v gets peeled?



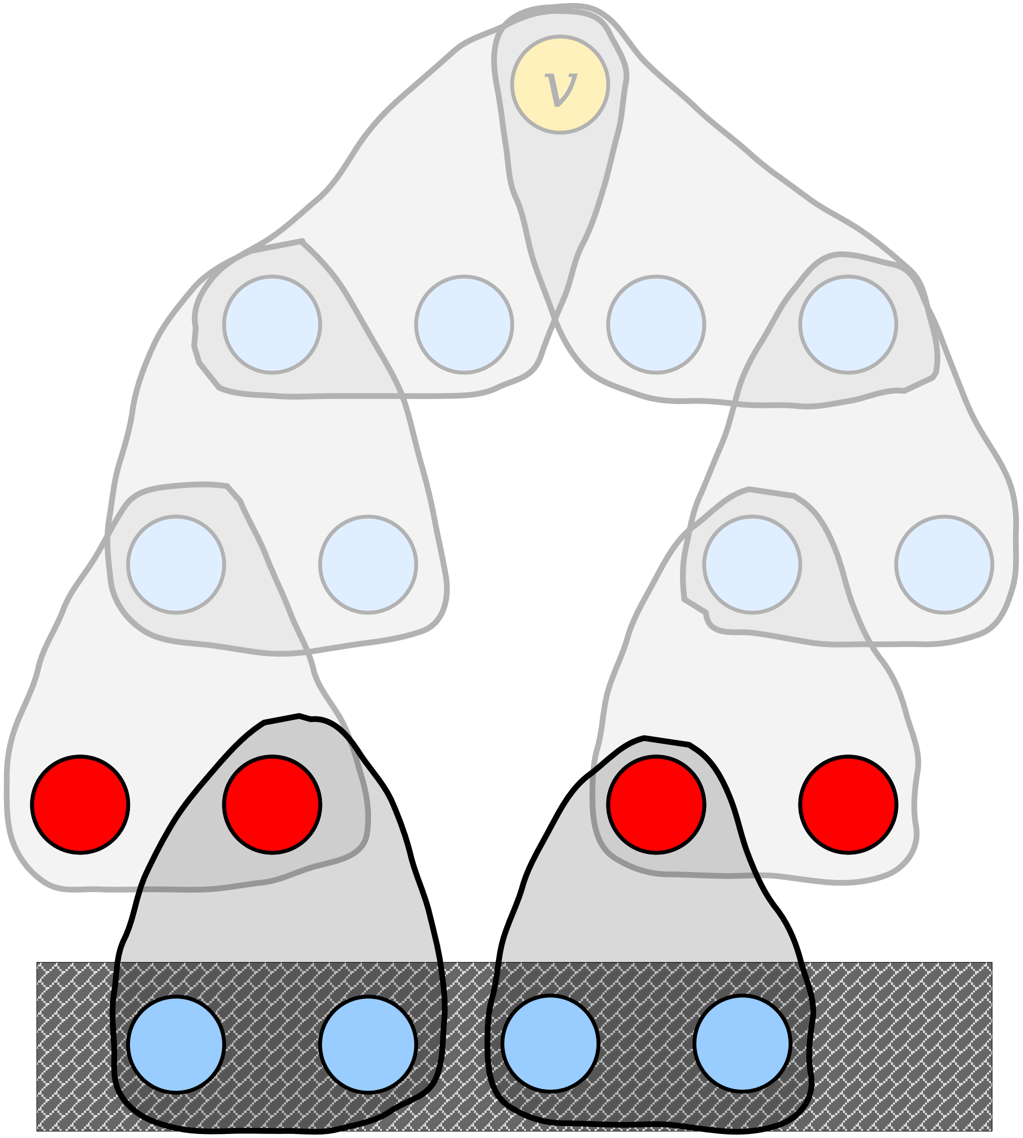
Bottom-Up Peeling

- To simplify the analysis, we'll look at a restricted form of peeling.
- We will assume all nodes at distance *r* from *v* are not peeled. (We don't know how they're linked up.)
- We'll then peel all peelable nodes at distance *r* – 1, then *r* – 2, then *r* – 3, …, then 1, and finally (if applicable) *v* itself.



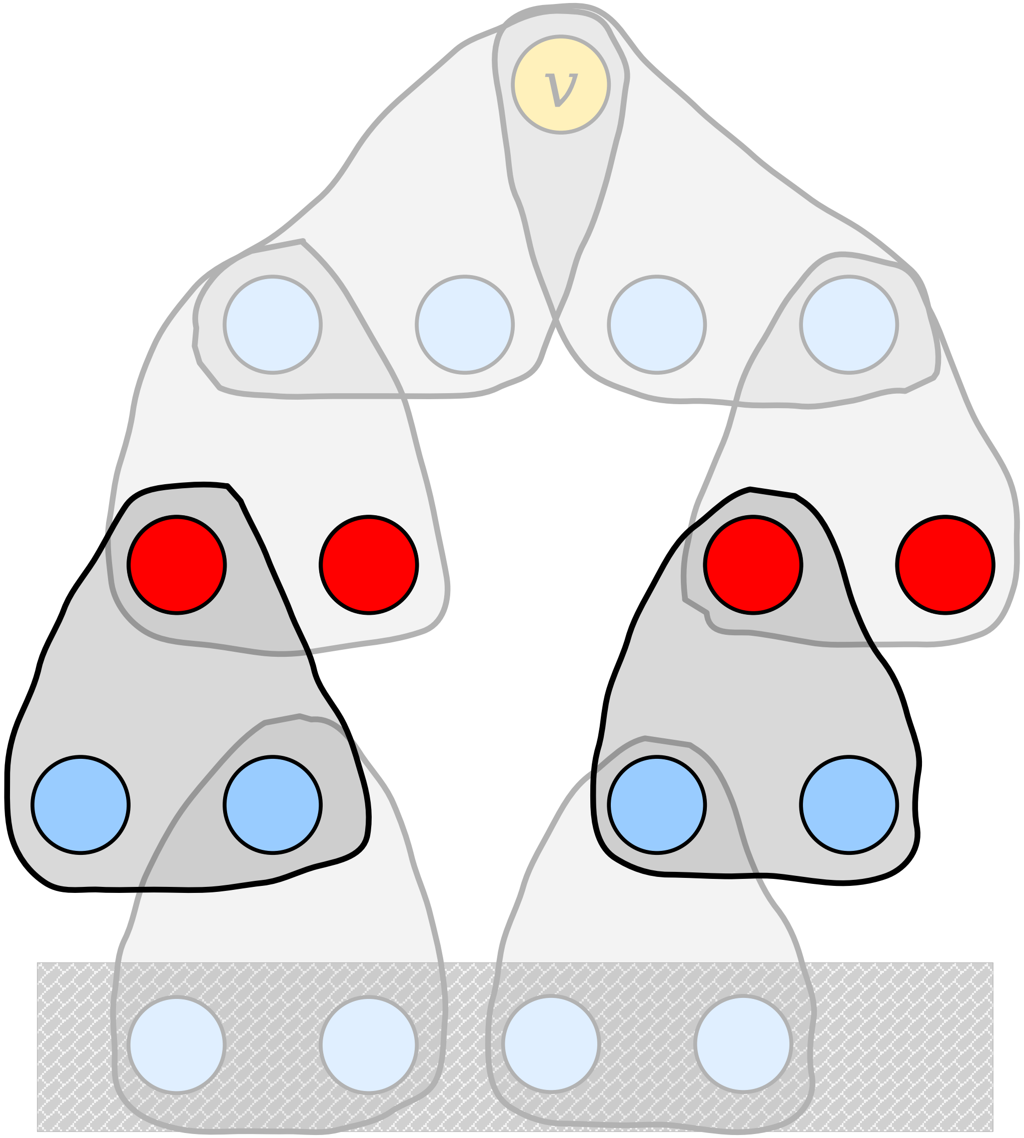
Bottom-Up Peeling

- Pick a node x in the penultimate layer of the tree. What's the probability x is peeled?
  - x has degree at least one (the edge back to its parent).
  - We never peel its children.
  - Thus it's peeled only in the case where it has no children.
  - Its child edge count is distributed as a Poisson($d\alpha$) variable.
- So Pr[x peeled] = $\boldsymbol{e^{-d\alpha}}$.



Bottom-Up Peeling

- Now look at a node x in the layer just above the previous one. What's the probability that x is peeled?
- ***Claim:*** The probability is ***not*** the same as for the row below this.
  - Nodes in the next row are peeled only if they have no children.
  - Nodes in this row are peeled if they have no children ***after the first round of peeling***.



Bottom-Up Peeling

- ***Claim:*** After one peeling round, x has

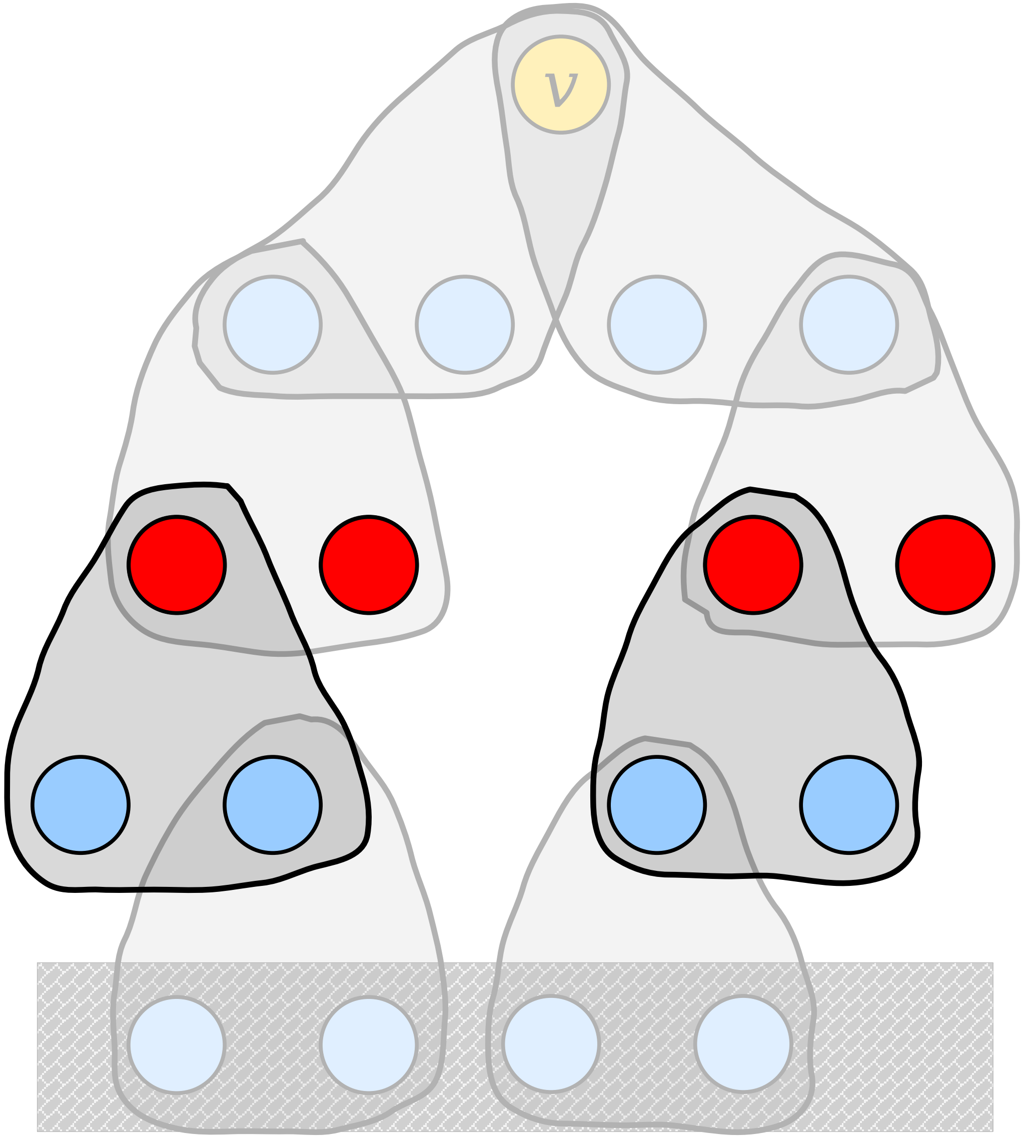
  $$\text{Poisson}(\boldsymbol{\alpha d}\mathbf{(1 - }\boldsymbol{p_1}\mathbf{)}^{\boldsymbol{d}\mathbf{-1}})$$

  child edges, where p_1 is the probability any individual node in the next layer was peeled.

  - Originally, x's child count was a Poisson($\boldsymbol{\alpha d}$) variable.
  - Each child hyperedge involves $\boldsymbol{d}$ **– 1** other nodes: d total nodes, minus one for x itself.
  - Each child edge remains iff none of those d – 1 nodes was peeled: probability $\mathbf{(1 - }\boldsymbol{p_1}\mathbf{)}^{\boldsymbol{d}\mathbf{-1}}$.

- Pr[x is peeled] =

  $$\boldsymbol{exp(-\alpha d(1 - p_1)^{d-1})}.$$



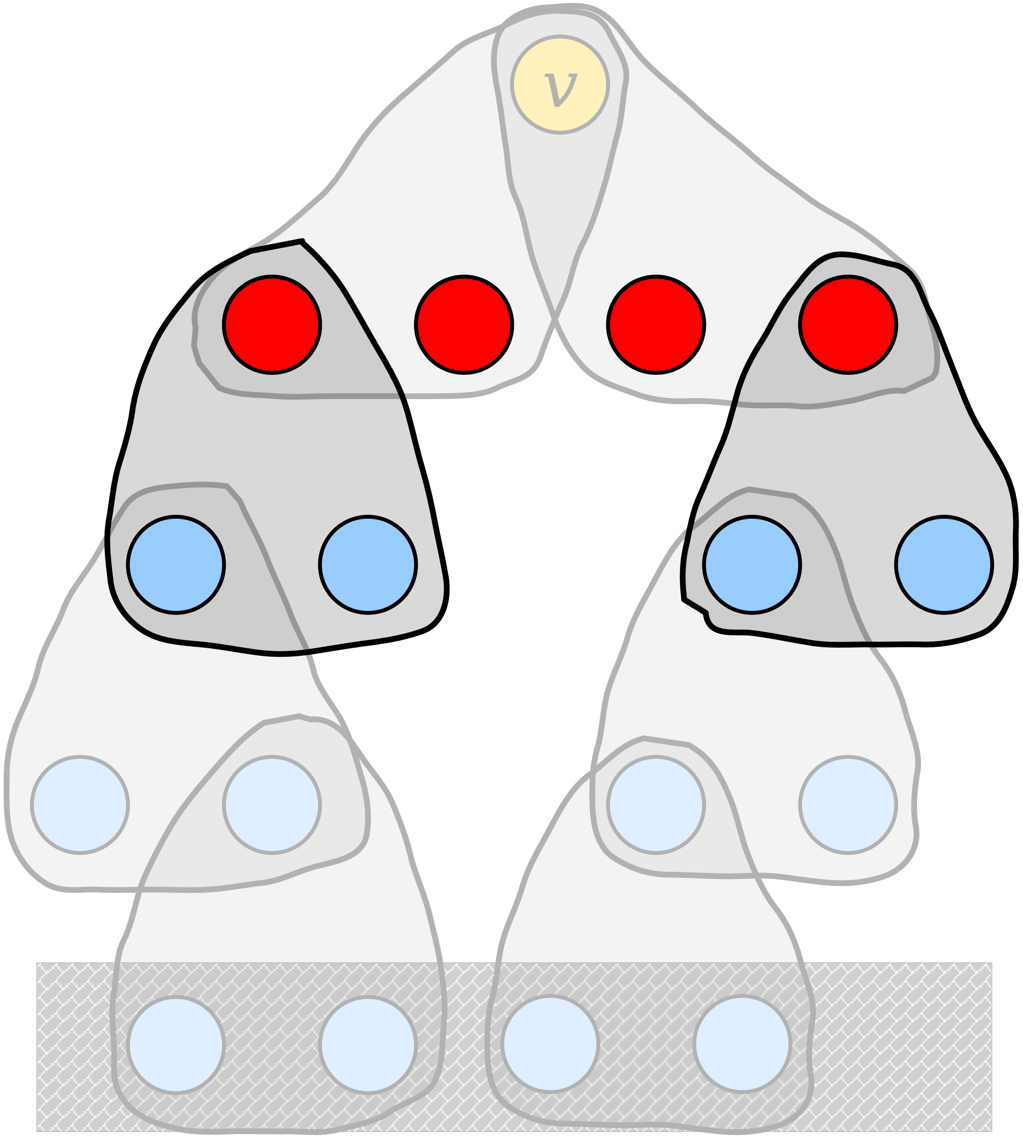
Bottom-Up Peeling

- Now look at a node *x* one level higher than before.
- Via analogous reasoning to the previous case, its unpeeled child edge count is a

  Poisson($\alpha d(1 - p_2)^{d-1}$)

  variable, where p_2 is the probability any individual node in the layer below it is peeled.
- Pr[*x* peeled] =

  $\exp(-\alpha d(1 - p_2)^{d-1})$.



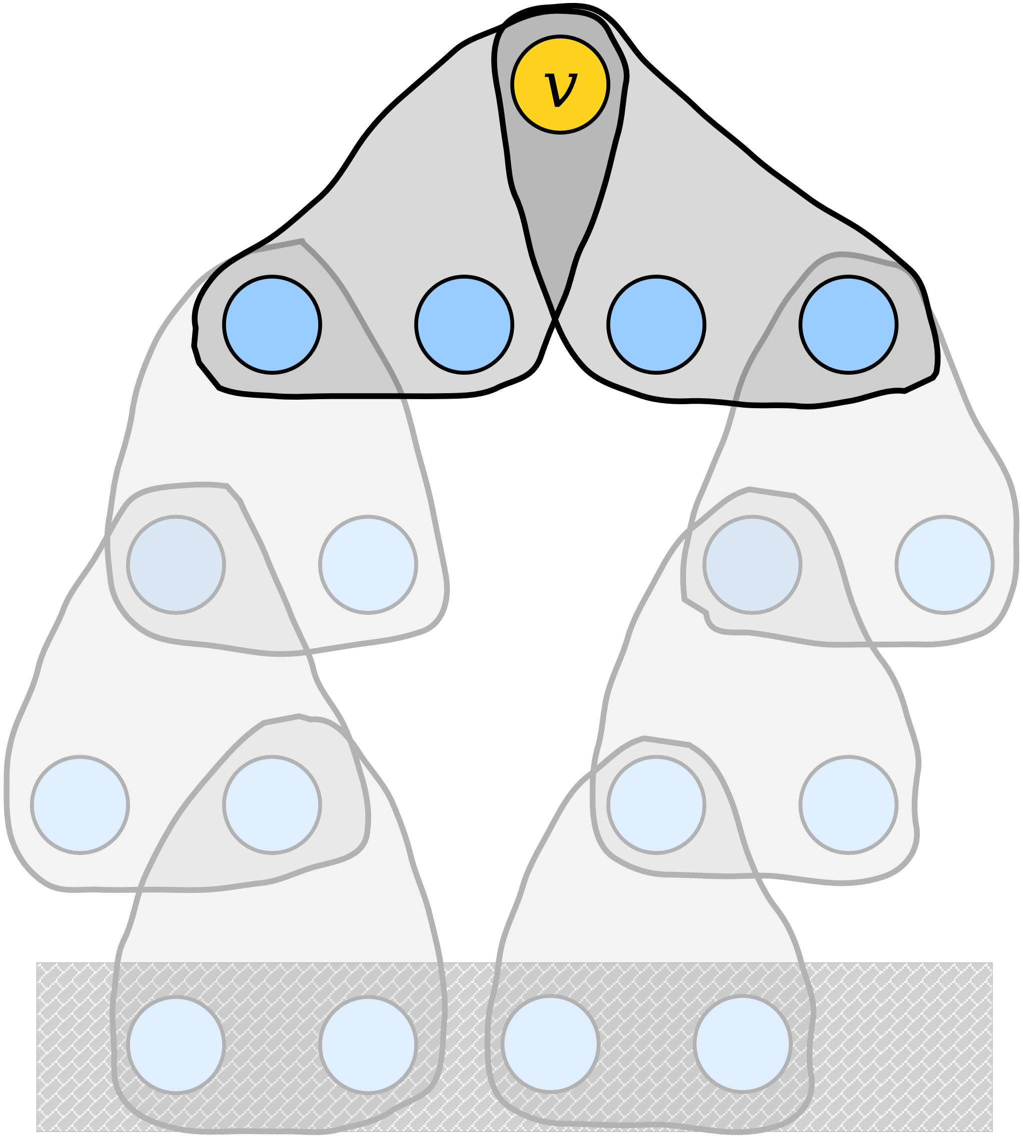
Bottom-Up Peeling

- Finally, look at *v* itself.
- Via analogous reasoning to the previous case, its unpeeled child edge count is a

  $$\textbf{Poisson}(\alpha d(1 - p_3)^{d-1})$$

  variable, where p_3 is the probability any individual node in the layer below it is peeled.
- Unlike other nodes in the tree, this node has no parent. Thus it's peeled if it has zero *or one* child.
- ***Claim:*** The one-child case makes a negligible contribution to the probability. (We'll substantiate this later.)
- Pr[*v* peeled] ≈

  $$exp(-\alpha d(1 - p_3)^{d-1})$$



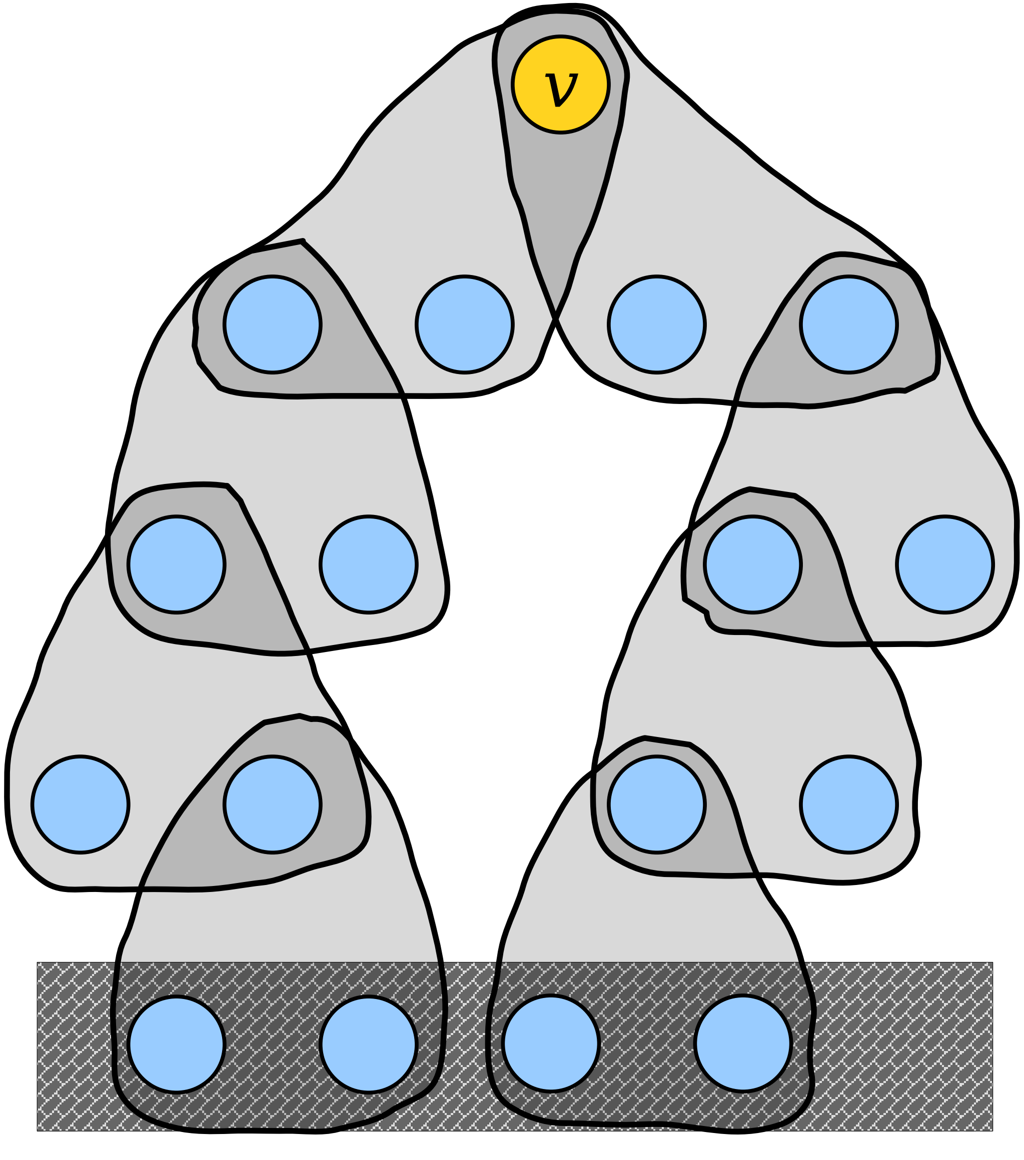
Bottom-Up Peeling

- More generally, let p_i be the probability that a node i layers from the bottom is peeled.
- We have the following recurrence relation:

$$p_0 = 0$$

$$p_{i+1} = e^{-\alpha d (1 - p_i)^{d-1}}$$

- How does this recurrence behave?
- Let's run some simulations!



Summarizing our Simulations

- For any fixed d, there seems to be some fixed threshold α_d* where
  - when $\alpha < \alpha_d$*, the probability of peeling the node rapidly approaches 1; and
  - when $\alpha > \alpha_d$*, the probability of peeling the node rapidly approaches a constant less than 1.
- This (plus some final technical details) explains why we're seeing the phase transition:
  - If $\alpha < \alpha_d$*, with high probability all nodes are peeled.
  - If $\alpha > \alpha_d$*, with high probability many nodes aren't.
- ***But why does this α_d* exist in the first place?***

Kleene’s Fixed-Point Theorem

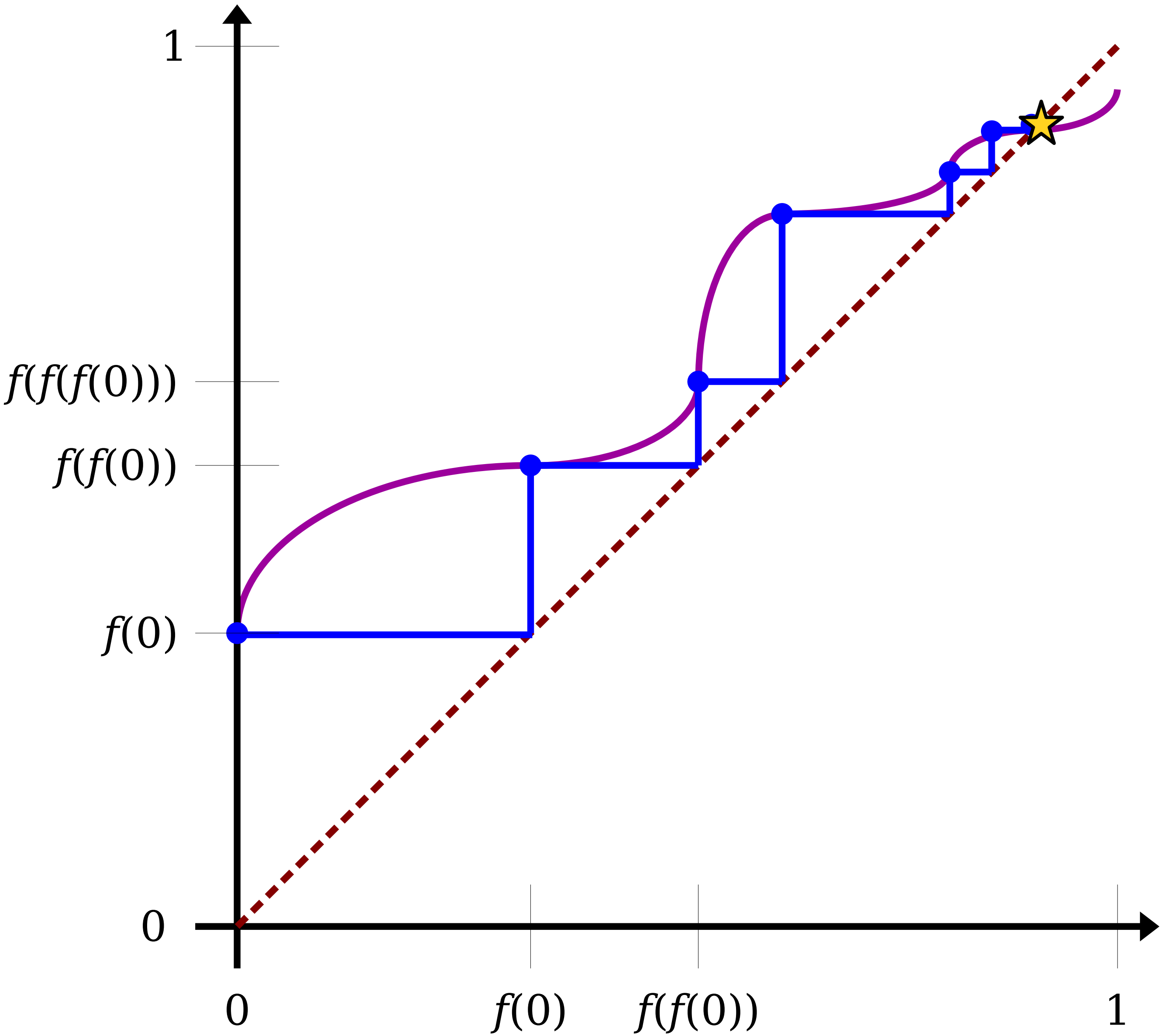
Iterated Functions

- Let $f(p) = exp(\text{-}\alpha d(1 - p)^{d-1})$.
- We're looking at the limit of this sequence:

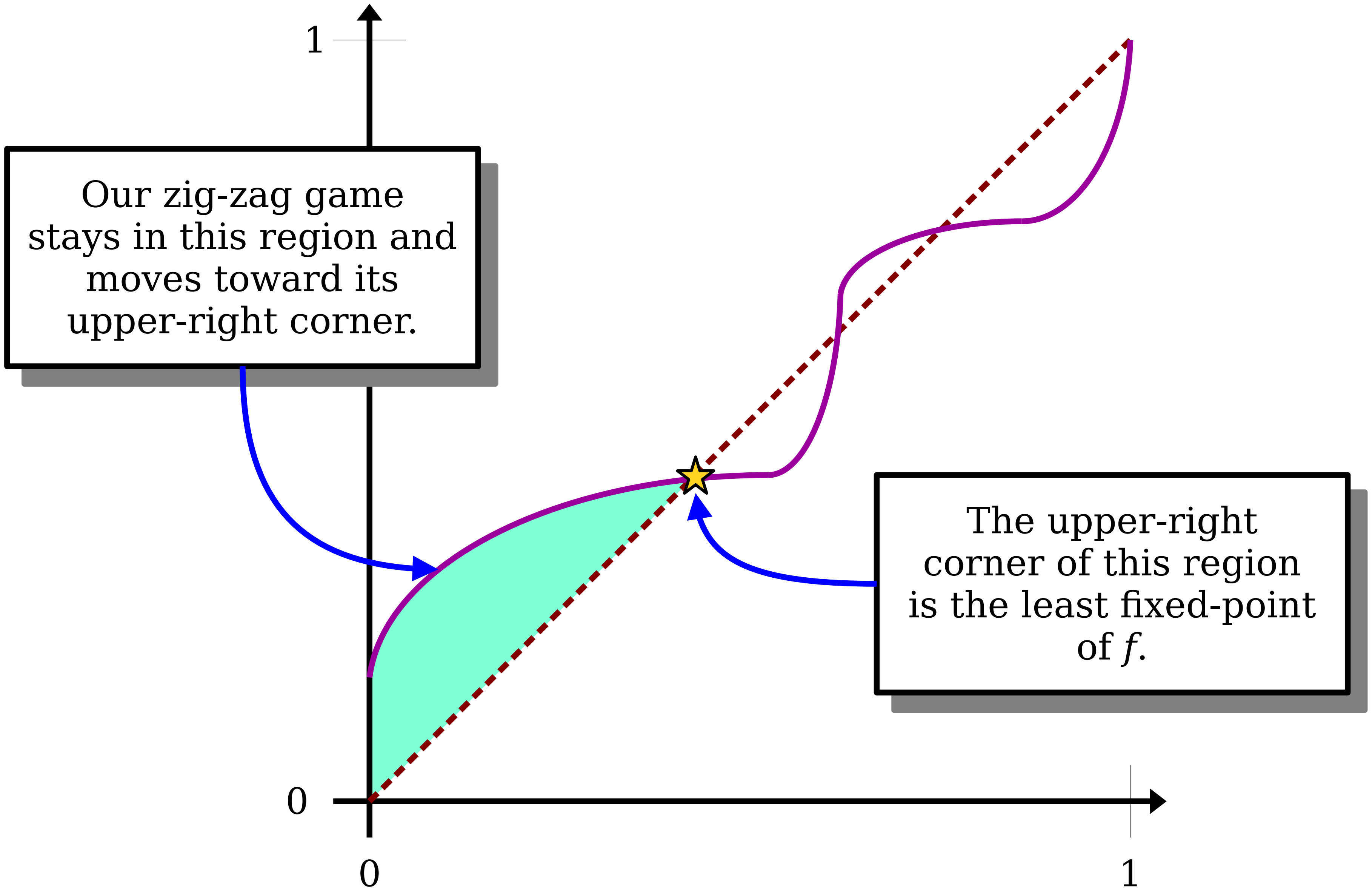
$$0, \quad f(0), \quad f(f(0)), \quad f(f(f(0))), \quad \ldots$$

- What happens if you iterate the same function lots and lots of times?

$$\boxed{\begin{aligned} p_0 &= 0 \\ p_{i+1} &= \boldsymbol{f(p_i)} \end{aligned}}$$



Let $f : [0, 1] \to [0, 1]$ be a continuous, monotone function. Then the sequence $0, f(0), f(f(0)), f(f(f(0))), \ldots$ converges to the smallest x for which $f(x) = x$.



Our zig-zag game stays in this region and moves toward its upper-right corner.

The upper-right corner of this region is the least fixed-point of *f*.

Let $f : [0, 1] \to [0, 1]$ be a continuous, monotone function. Then the sequence $0, f(0), f(f(0)), f(f(f(0))), \ldots$ converges to the smallest x for which $f(x) = x$.

Something ***very interesting*** happens
if we apply Kleene’s Fixed-Point Theorem
to our particular recurrence…

Putting it All Together

- The probability that a node gets peeled increases the more layers we peel back, according to the transform

$$p \mapsto exp(-\alpha d(1 - p)^{d - 1}).$$

- This converges to the least fixed point of this map.
- This curve always has a fixed point of $p = 1$, but for each d there's an α_d* where the curve gains a second fixed point below 1 as it intersects the line $y = x$.
- When $\alpha < \alpha_d$*, we converge to the fixed point of 1 and peeling almost always succeeds.
- When $\alpha > \alpha_d$*, we converge to the lower fixed point and peeling has only a constant (non-1) probability of success.

Recap from Today

- IBTLs require $\approx 1.23k$ buckets to be communicated to solve set reconciliation when the two sets differ in k positions.
- Hypergraph peeling is a powerful technique for packing a lot of retrievable information into a small space.
- Hypergraph peeling has a phase transition due to the behavior of an iterated system of functions converging to a least fixed point.

Next Time

- ***Bloom Filters***
  - Saving valuable time by eliminating unnecessary work, most of the time.
- ***… Are Now Obsolete***
  - And the replacements are *amazing*.